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Mathematical Model of the Cylinder Rotations in a Viscous Medium

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Abstract. In the paper movement of a rotating cylinder in a viscous medium is considered. Control law provides given position of the cylinder at the finite time minimizing the work of the drag forces to rotation is found. In the case when the control force is applied in the direction of the axis of symmetry the problem is singular. So classical variational procedure for solving this task does not lead to success.

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INTRODUCTION

In the problems of energy optimization [6] Euler–Lagrange equations do not contain control explicitly. This is an indication that the optimal control force contains impulse components [4]. In this situation, the problem of optimization of control forces may be reducing the problem minimizing of the drag forces, taking into account the kinematic relations. Optimal solution of the auxiliary problem will not include impulse components [5]. And such solution can be found using the classical variational procedure of Euler–Lagrange.

MATHEMATICAL MODEL

Problem statement

In the framework of problem the motion of a rotating cylinder in a viscous medium is considered. It is obvious [1] at the initial time phase coordinates of the rotating cylinder have the following values

$$x(0) = 0, \quad \dot{x}(0) = 0, \quad \varphi(0) = 0, \quad \dot{\varphi}(0) = \omega_0. \tag{1}$$

Here x and φ are linear and angular generalized coordinates. Equations of motion have the form

$$m\ddot{x} = F, \quad J\ddot{\varphi} = -L, \tag{2}$$

where m, J – mass and moment of inertia of the cylinder, F – control force acting along the axis of symmetry, L – full drag forces moment of rotation [2]. At the terminal time t_k is required that the cylinder was at a given distance

$$x(t_k) = x_k. \tag{3}$$

For the elementary time interval dt drag force does work

$$dA = Ld\varphi = L\omega dt\ . \tag{4}$$

Hence the expression for the power of drag force

$$\dot{Y} = \omega L\ , \tag{5}$$

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which is considered as a differential equation for work $Y(t)$ done by the drag forces of rotation at the time t_k with the initial condition $Y(0) = 0$.

Now the following dynamic optimization problem can be considered.

Problem 1. It is required to find control $F(t)$ which solves the problem of minimizing

$$Y(t_k) \to \min$$

with dynamic constraints (2), (5) and boundary condition (3).

At time t_k the following formula holds

$$Y(t_k) = -J\omega^2(t_k)/2 + J\omega^2(0)/2 \,. \tag{6}$$

Lemma 1. Problem 1 is equivalent to maximizing the angular velocity of rotation of the cylinder at the last moment control process.

Meaning of this assertion is that it allows one to solve the Problem 1 does not take into account the dynamic relation (5), which reduces the dimension of the problem by one.

The system of differential equations (2) in the normal form of Cauchy has the form

$$\dot{x} = U, \quad \dot{U} = F/m, \quad \dot{\varphi} = \omega, \quad \dot{\omega} = -L/J \,. \tag{7}$$

Because the Problem 1 is no limit the value of the angular variable in the last moment of the control process then the third equation of system (7) can be removed. As a result, the Problem 1 becomes equivalent to the following

$$\begin{aligned} &\omega(t_k) \to \max, \\ &\dot{x} = U, \qquad x(0) = 0, \quad x(t_k) = x_k, \\ &\dot{U} = F/m, \quad U(0) = 0, \\ &\dot{\omega} = -L/J, \quad \omega(0) = 0. \end{aligned} \tag{8}$$

In order to have a formal possibility to apply the variational procedure Euler–Lagrange to the problem (8) we need in the following constraint: full moment of drag forces of cylinder rotation is determined by its current phase coordinates [8].

Thus, further for finding the optimal mode of the cylinder motion one can apply the classical procedure of the Euler–Lagrange. Namely it is necessary to write the Hamiltonian

$$H = \lambda_1 U + \lambda_2 F/m - \lambda_3 L/J,$$

where λ_i – Lagrange multipliers, $i = 1,2,3$, and find the optimal control F as the solution of the Euler–Lagrange equation

$$\partial H/\partial F = 0.$$

In this case it is reduced only to the next request $\lambda_2 = 0$.

So the problem indeed is singular dynamic optimization problem. And in the optimal control force in addition to continuous component composes impulse components.

The reduction of the problem

Investigated the Problem 1 can be reduced to the problem of maximizing the angular velocity in the terminal time taking into account only the kinematic relations describing the motion of the cylinder. The optimal solution of the auxiliary problem does not contain impulse components.

It is possible to raise the following problem.

Problem 2. Find the program changes the linear velocity $U(t)$, which solves the problem of maximizing

$$\begin{aligned} &\omega(t_k) \to \max, \\ &\dot{x} = U, \qquad x(0) = 0, \qquad x(t_k) = x_k, \\ &\dot{\omega} = -LJ^{-1}, \quad \omega(0) = \omega_0. \end{aligned} \tag{9}$$

Solution of the auxiliary problem

To solve the Problem 2 one can use the the classic procedure of Euler–Lagrange. To do this, make the Hamiltonian

$$H = \lambda_1 U - \lambda_2 J^{-1} L, \tag{10}$$

conjugated system

$$\begin{aligned} -\dot{\lambda}_1 &= \partial H/\partial x = 0, \\ -\dot{\lambda}_2 &= \partial H/\partial \omega = -\lambda_2 J^{-1}\,\partial L/\partial \omega, \end{aligned} \tag{11}$$

and the functional

$$\Phi = \nu(x(t_k) - x_k) - \omega(t_k)$$

to find the boundary values

$$\lambda_1(t_k) = \partial\Phi/\partial x(t_k) = \nu, \quad \lambda_2(t_k) = \partial\Phi/\partial\omega(t_k) = -2\omega(t_k). \tag{12}$$

Here ν – Lagrange multiplier and its purpose is providing a boundary condition for distance.

From (10) – (12) it follows that

$$\lambda_1 \equiv \nu, \tag{13}$$

$$\dot{\lambda}_2 = \lambda_2 J^{-1}\,\partial L/\partial\omega, \quad \lambda_2(t_k) = -2\omega(t_k). \tag{14}$$

Substituting (13) in (10) leads to the expression

$$H = (\omega - \lambda_2 J^{-1})L + \nu U. \tag{15}$$

Optimal control must satisfy to the equation of Euler–Lagrange

$$0 = \partial H/\partial U = \nu - \lambda_2 J^{-1}\,\partial L/\partial U. \tag{16}$$

Because the dynamics in the Problem 2 are stationary then in the process of optimal control Hamiltonian is constant, that is

$$H = (\omega - \lambda_2 J^{-1})L + \nu U = const. \tag{17}$$

Solving (16) with respect to $\lambda_2 J^{-1}$ and substituting the result in (17) one can get the final relation

$$U - L/L_U = const = -k_0, \tag{18}$$

where $L_U = \partial L/\partial U$.

OPTIMAL SOLUTION FOR THE CASE CONSIDERED BY N. A. SLEZKIN

Formula for the total moment of drag forces of the cylinder rotation by the flow parallel to its axis [3] is

$$L = C_f \pi a^3 \rho l \omega \sqrt{U^2 + a^2\omega^2}, \tag{19}$$

where a and l – radius and length of the cylinder respectively, ρ – flow density, C_f – drag coefficient depends on the Reynolds number

$$C_f = \frac{\tau}{\sqrt{\mathrm{Re}}}, \quad \tau \approx 1.328, \quad \mathrm{Re} = \frac{Ul}{\nu}, \tag{20}$$

ν – kinematic viscosity. Thus the angular momentum of the drag forces of the cylinder rotation depends not only on the angular velocity of rotation of ω, but also on the velocity U linear flow.

According to formulas (19), (20) expression for the angular momentum of the cylinder can be written as

$$L = \tau\pi a^3 \sqrt{\nu l}\,\rho\omega\sqrt{U + \frac{a^2\omega^2}{U}}. \tag{21}$$

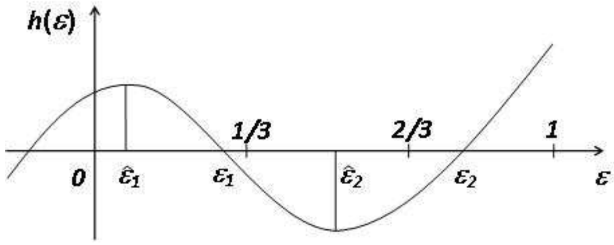

FIGURE 1. Zeros of function $h(\varepsilon, c)$

Inertia Momentum of the cylinder is calculated by the formula

$$J = \frac{1}{2}\pi a^4 l \rho_c,$$

where ρ_c – the average mass density of the cylinder. Now the equations of cylinder motion (9) take the form

$$\begin{aligned} &\dot{x} = U, \;\; x(0) = 0, \;\; x(t_k) = x_k, \\ &\dot{\omega} = -\frac{2\tau}{a}\sqrt{\frac{\nu}{l}}\,\frac{\rho}{\rho_c}\,\omega\sqrt{U + \frac{a^2\omega^2}{U}}, \;\; \omega(0) = \omega_0. \end{aligned} \tag{22}$$

Calculation of partial derivative of (21) on the linear velocity of the cylinder leads to the expression

$$L_U = 2\tau\pi a^3\sqrt{\nu l}\,\rho\,\omega\sqrt{U + \frac{a^2\omega^2}{U}}\left(1 - \frac{a^2\omega^2}{U^2}\right)^{-1}. \tag{23}$$

From formulas (21) and (23) follows

$$\frac{L}{L_U} = 2\,\frac{U^3 + Ua^2\omega^2}{U^2 - a^2\omega^2}.$$

Substituting this expression in (ref 2.3.20) gives an algebraic equation of the third degree with respect to the current value of the optimal linear velocity of the cylinder

$$U^2(U - k_0) + a^2\omega^2(3U + k_0) = 0. \tag{24}$$

So the Problem 2 in this case is reduced to a boundary value problem for system (22), in which control is one of the roots of the equation (24): choose k_0 so as to provide at the last time of control process the given distance moving cylinder.

NUMERICAL MODELING OF THE OPTIMAL TRAJECTORY

Modeling optimal displacement cylinder connected with the search of the roots of the equation (24). Difficulty arises here due to the fact that the range of roots is not known [7]. However, this difficulty can be overcome as follows. The solution of equation (24) conveniently sought as

$$U = k_0\varepsilon. \tag{25}$$

Then ε is one of the roots of the equation

$$\varepsilon^2(\varepsilon - 1) + c^2(3\varepsilon + 1) = 0, \tag{26}$$

where

$$c = a\omega/k_0. \tag{27}$$

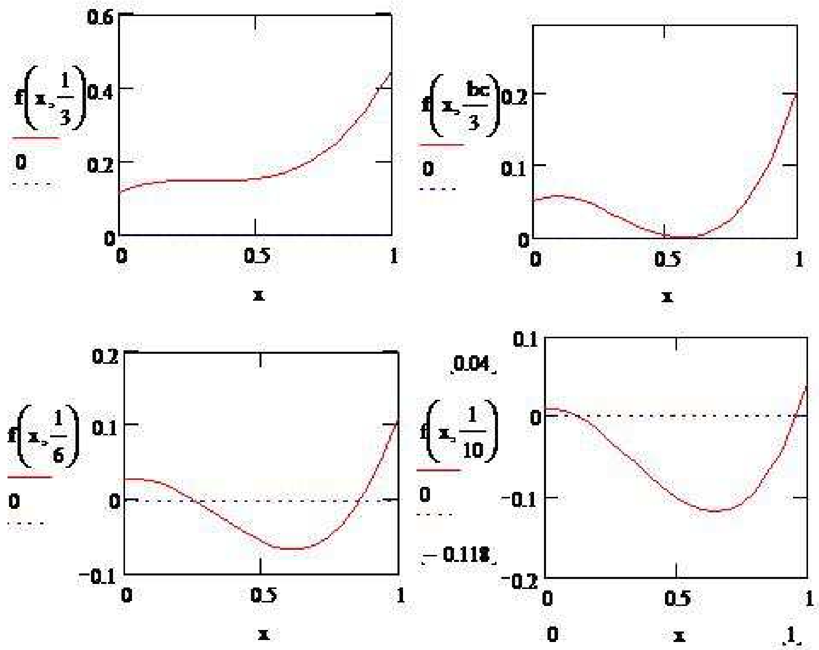

FIGURE 2. Zeros of function $h(\varepsilon, c)$ with a decrease c

TABLE 1. Data of numerical simulation

a, m	l, m	$\nu, m^2/sec$	ω_0, sec^{-1}	t_k, sec	x_k, m
$0,3$	$2,4$	$1,13 \cdot 10^{-4}$	25	$2,21$	11000

The magnitude of c^2 has a simple physical meaning: it is the ratio of the kinetic energy of rotation of the cylinder to the kinetic energy of its translational motion with velocity k_0.

Analysis of the equation (26) indicates that the parameter ε satisfies to the constraint

$$0 < \varepsilon < 1, \tag{28}$$

if only constant $k_0 > 0$. Last, according to (24) means demand: at an optimal move like N. A. Slezkin $U > a\omega$. Condition (28) significantly facilitates search for the roots of equation (26). Here the standard numerical methods can be used.

The left side of equation (26) is a function $h(\varepsilon, c)$ variables ε and c. Mutual disposition of zeros h and its derivative with respect to ε is shown in the Fig. 1. Moreover, with increasing time the root $\varepsilon_1(t)$ and the value $\hat{\varepsilon}_1(t)$ are reduced, and the root $\varepsilon_2(t)$ and the value $\hat{\varepsilon}_2(t)$ increased.

Mutual disposition of zeros function h with a decrease in the $c = a\omega/k_0$ shown in the Fig. 2.

Numerical experiment (see Tab. 1) led to the conclusion that in the formation of optimal programs changes linear motion speed of the cylinder is always involved larger root of the equation (26). The reason for this outcome is that, although the control corresponding to the root of ε_2 and worst slows down the rotational motion of the cylinder, but provides the best linear speed of its movement.

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