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Evaluation of Payment Flows Based on Markov Chain Model with Incomplete Information

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Abstract. The estimation problem of the loan portfolio profitability is considered. The payment flows generated by the portfolio are modeled as a Markov process with incomplete information. The loans in the portfolio are divided into several groups depending on presence of indebtedness and its terms. It is proposed that dynamics of portfolio shares is described by the Markov chain model with discrete time and incompletely known transition probabilities. The simulation modeling and the confidence approach are used to evaluate the profitability of the portfolio.

Keywords: Markov Chain, Incomplete Information, Control, Loan Portfolio, Profitability
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INTRODUCTION

Markov chains are used in finance and economics to model a variety of different phenomena, including asset prices and market crashes.

We consider the forecasting problem for profitability of a bank's credit portfolio. Changes in the portfolio are described by a Markov random process with discrete time and finite number of states. The Markov chain model is usually used to describe the loan portfolio dynamics [3, 7, 8]. By the state of a loan we mean that it belongs to a certain group of loans with respect to the existence and duration of arrears. The matrix of transitional probabilities is not known exactly, and information about it is collected during the system's operation. The methods of forecast the portfolio risk and calculate necessary reserves were proposed in [9, 10]. The portfolio risk is set as a share of problematic loans.

A problem of the cash flows evaluation for the loan portfolio is considered in the paper. We assume that the portfolio profitability is described by the expected Net Present Value (*NPV*). The scheme with repayment is studied to forecast profitability of the portfolio for the full cycle of life. The simulation modeling and the confidence approach are proposed for estimation of the portfolio profitability.

MATHEMATICAL MODEL OF THE LOAN PORTFOLIO

Credit portfolio shares

A change of shares of credits portfolio is described by Markov chain with discrete time. The credit state is determined on as an accessory to some group of credits depending on presence of indebtedness and its terms. The process starts in one of these states and moves successively from one state to another.

The shares $x_1,\dots,x_k$ of the portfolio may be defined by two ways:

1) as a ratio of the number of loans:

$$x_i = \frac{n_i}{n}, \tag{1}$$

where n_i is the number of credits in i-th groups, $n = n_1 + \dots + n_k$.

2) as a ratio of the remaining principal amounts:

$$\tilde{x}_i = \frac{A_i}{A}, \tag{2}$$

where A_i is the sum of principal amounts for all contracts in i-th group, $A = A_1 + \dots + A_k$.

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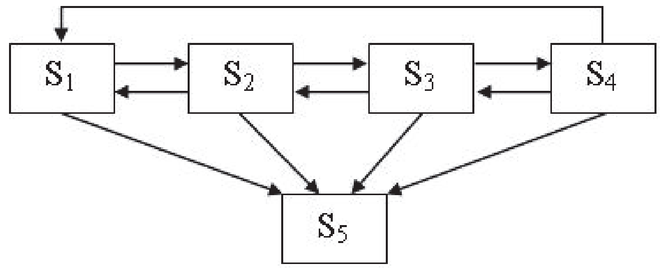

FIGURE 1. Graph of the system states

The first approach is investigated better, but the second one is often used by managers analyzing loan portfolios.

We consider a scheme with repayment. In practice, such scheme is used for evaluating the purchased portfolio, which is not recharging by new loans. The goal of the analysis is the forecast of the portfolio profitability for the full cycle of its life.

For example, let consider a simple scheme with 5 credit groups and repayment:

1. loans without delay, including new (S_1);
2. loans with delay less than 31 days (S_2);
3. loans with delay more than 30 days but less than 61 days (S_3);
4. loans with delay more than 60 days – problematic loans (S_4);
5. repayed loans (S_5).

Dynamics of the Portfolio Shares

Let $\xi_t = \{\xi(0), \ldots, \xi(t)\}$ be a discrete Markov chain with k states. Denote the probability that the system is in i-th state at the moment t by $x_i(t)$, $i = \overline{1,k}$:

$$x_i(t) = \mathscr{P}\{\xi(t) = e_i\}, \;\; i = \overline{1,k},$$

where $\{e_1, \ldots, e_k\}$ is the unit basis in $\mathbf{R}^k$, $\mathscr{P}(A)$ is the probability of a random event A.

The following conditions hold:

$$0 \le x_i(t) \le 1, \quad x_1(t) + \ldots + x_k(t) = 1.$$

Denote by p_{ij} a probability of going from state i to state j on one step

$$\mathscr{P}\{\xi(t+1) = e_j | \xi(t) = e_i\} = p_{ij}.$$

It is supposed that these probabilities are constant $p_{ij}(t) \equiv p_{ij}$, $i = \overline{1,k}$, $j = \overline{1,k}$. That is we consider a stable economic situation.

Denote by $x(t) = \{x_1(t), \ldots, x_k(t)\}^\top$ the vector of states probabilities, and $k \times k$-matrix of the transition probabilities by $P = \{p_{ij}\}$. The dynamics of the system states probabilities is described by the difference equation [6]:

$$x(t+1) = P^\top x(t), \quad t = 0, 1, \ldots, T. \tag{3}$$

The portfolio risk is usually defined as the share of problematic loans [3, 7]. In case of incompletely known matrix $P \in \mathbf{P}$ the portfolio risk forecast bases on the estimation of the system state [10].

For scheme with repayment (e.g. for scheme in Fig.1) all shares of the portfolio tend to zero except the share of repayment loans:

$$\lim_{t\to\infty} x_i(t) = 0, \;\; i = \overline{1,k-1}, \;\; \lim_{t\to\infty} x_k(t) = 1.$$

It results from the presence of the absorbing state S_k and properties of the absorbing Markov chain [6].

PROFITABILITY OF A PROJECT

Net Present Value

Let a state of some investment project is described by the discrete Markov chain $\xi_t = \{\xi(1), \dots, \xi(t)\}$. Denote the income from the project at time t by $A(t) = A(t, \xi_t)$, $A(0)$ is a known negative value.

We propose the profitability of the project is described by the mean value of the Net Present Value (NPV), where

$$NPV(T) = NPV(T, \xi_T) = A(0) + \sum_{t=1}^{T} \frac{A(t, \xi_t)}{(1+r)^t}, \tag{4}$$

r is the discount rate (risk-free interest rate for one period).

Let us consider a random process $B(T, \xi_t)$:

$$B(t+1, \xi_{t+1}) = B(t, \xi_t) + a(t+1)A(t+1, \xi_{t+1}),\ B(0) = A(0). \tag{5}$$

If we defined

$$a(t+1) = \frac{a(t)}{1+r(t)}, \quad a(0) = 1, \tag{6}$$

then $NPV(T, \xi_T)$ is equal to $B(T, \xi_T)$.

Payment Flow of a Contract

Consider the payment flow of a loan contract with the initial sum $D(0)$ and the annuity payments d:

$$d = k_b D(0), \quad k_b = \frac{b(1+b)^\tau}{(1+b)^\tau - 1}, \tag{7}$$

here b is a monthly rate of the contract, τ is a term of the contract. Denote by $D(t, \xi_t)$ the principal amount of the contract at the moment t.

In the Markov model a payment $A(t, \xi_t)$ at the period $[t; t+1]$ is a random value depended on the numbers of groups i and j, where the loan was in i-th group at moment t and goes to the j-th group at $t+1$.

Let's describe a payment flow for an arbitrary contract.

- If the annuity payment $d = k_b D(0)$ was paid then the principal amount decreased

  $$A(t+1, \xi_{t+1}) = k_b D(0), \quad D(t+1, \xi_{t+1}) = (1+b)D(t, \xi_t) - k_b D(0),$$

  and the group number did not change on the step $t \to t+1$ (except a case of a credit from problematic group). Here k_b is defined by equation (7).
- If there was no payment on the contract in the last month then

  $$A(t+1, \xi_{t+1}) = 0,\ D(t+1, \xi_{t+1}) = (1+b)D(t, \xi_t),$$

  and the group number increased by 1 or it did not change for the problematic group.
- If there were two annuity payments on a contract in last month then the group number decreased by 1. And so on.
- In case of the repayment of the whole contract at moment $t+1$ the loan goes to the last group of repayed loans and

  $$A(t+1, \xi_{t+1}) = (1+b)D(t, \xi_t),\ D(t+1, \xi_{t+1}) = 0.$$

Thus we get stochastic difference equations for the payment flow $A(t, \xi_t)$ and the principal amount $D(t, \xi_t)$:

$$A(t+1, \xi_{t+1}) = \xi(t)^\top (d \cdot C_1 + (1+b)D(t, \xi_t)C_2)\xi(t+1),\ t = 0, \dots, T, \tag{8}$$

$$D(t+1, \xi_{t+1}) = (1+b)D(t, \xi_t) - \xi(t)^\top (d \cdot C_1 + (1+b)D(t, \xi_t)C_2)\xi(t+1), \tag{9}$$

where

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 4 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(0) = -D(0).$$

Portfolio Net Present Value

If the transition probability matrix P is known we can estimate expected Net Present value of the loan portfolio $\mathscr{E}(NPV) = \overline{NPV}$ by calculation the mean value of the $B(T,\xi_T)$, where $B(t,\xi_t)$ is defined by equations (5),(8),(9), ξ_t is the Markov chain with the transition probability matrix P.

Expected profitability $\overline{NPV}$ depends on a sum of principal amounts of all contract $D(0)$, initial structure of the portfolio $x(0)$, the risk-free interest rate r, the interest rate of the loans b and the matrix of transition probabilities P:

$$\overline{NPV}(T) = D(0)c(r,b,P,T), \quad c(r,b,P,T) = \mathscr{E}(B(\xi_T;r,b,P,T)).$$

Here $B(\xi_T;b,r,P,T)$ is the solution of the stochastic equations with the initial condition $B(0)=1$, $\mathscr{E}(B(\xi_T))$ is the mean value of $B(\xi_T)$.

If P and r are known we can calculate the expected portfolio profitability ($\overline{NPV}$) using the linear difference moments equations [11] for the linear multistage stochastic system depended on simple Markov chain. But transition probabilities p_{ij} as a rule are incompletely known and evaluated on the basis of statistical data.

ESTIMATION OF PORTFOLIO PROFITABILITY

Estimation of the Transition Probabilities

For the estimation of the probability p_{ij} one usually use the statistical data about transitions from one state to another. The probability of transition from any given state i is approximated by the ratio of the number of individuals that started in state i and ended in state j to the number of all individuals that started in state i:

$$w_{ij} = \frac{n_{ij}}{n_i}. \tag{10}$$

Anderson and Goodman [1] showed that the estimates w_{ij} given by formulae (10) are approximately normal distributed and obtained the statistical moments of the estimates

$$\mathscr{E}(w_{ij}(t)) = p_{ij}, \;\; Var(w_{ij}(t)) = \frac{p_{ij}(1-p_{ij})}{n_i},$$

$$Cov(w_{ij}(t), w_{il}(t)) = -\frac{p_{ij}p_{il}}{n_i}, \;\; j \neq l, \quad Cov(w_{ij}(t), w_{cl}(t)) = 0, \;\; i \neq c. \tag{11}$$

Here $\mathscr{E}(\xi)$ is the mathematical expectation of a random value ξ, $Var(\xi)$ is its variance, $Cov(\xi,\zeta)$ is the covariance between ξ and ζ.

Confidence Estimation for Transition Probabilities

Let us consider confidence regions of a level β for an unknown row $p^{(i)} = \{p_{ij}, j = \overline{1,k}, j \neq i\}$ of the transition matrix P. They may be written as an ellipsoid

$$Z_\beta^{(i)} = \{p_{ij} \in \mathbf{R}_+^{k-1} : (p^{(i)} - w^{(i)})^\top G_i^{-1}(p^{(i)} - w^{(i)}) \leq \tau^{(\beta)}\}. \tag{12}$$

Here $w^{(i)} = \{w_{ij},\ j = \overline{1,k}\}$, G_i is the estimate of the covariance matrix

$$G_i = \frac{1}{n_i}\begin{pmatrix} w_{i1}(1-w_{i1}) & -w_{i1}w_{i2} & \dots & -w_{i1}w_{ik} \\ -w_{i2}w_{i1} & w_{i2}(1-w_{i2}) & \dots & -w_{i2}w_{ik} \\ \dots & \dots & \dots & \dots \\ -w_{ik}p_{i1} & -w_{ik}w_{i2} & \dots & w_{ik}(1-w_{ik}) \end{pmatrix},$$

$\tau^{(\beta)}$ is β-quantile for χ^2 distribution with $k-1$ degrees of freedom.

Another form of a confidence region is a parallelotop [4]. For the row $p^{(i)}$ it has the form

$$Z_\beta^{(i)} = w^{(i)} + \tilde{G}_i B_\beta \subset \mathbf{R}^{k-1}, \tag{13}$$

here $\tilde{G}_i \tilde{G}_i^\top = G_i$, B_β is the confidence region of the level β for standard Gaussian $(k-1)$-vector ζ:

$$B_\beta = \{b \in \mathbf{R}^{k-1} : |b_s| \le \tau_\gamma,\ \ s = \overline{1,k-1}\},$$

τ_γ is two-sided quantile of level γ for normal distribution, $\beta = \gamma^{k-1}$.

Let $\beta = \sqrt[k]{\alpha}$, we construct confidence regions $Z_\beta^{(i)}$ for transition probabilities vectors $p^{(i)}$. The sets $Z_\beta^{(i)}$ may be given as the ellipsoids (12) or as the parallelotops (13). A confidence region of level α for the probabilities $\{p_{ij},\ j \neq i\}$ has the form

$$\hat{Z}_\alpha = Z_\beta^{(1)} \times \ldots \times Z_\beta^{(k)}. \tag{14}$$

Estimates for p_{ii} follow from the equalities

$$p_{i1} + \ldots + p_{ik} = 1, \quad i = \overline{1,k}. \tag{15}$$

We denote by Z_α the confidence set for all elements of the matrix P constructed on the base $\hat{Z}_\alpha$ with the condition (15).

Thus, the algorithm for finding the confidence estimate of the portfolio profitability is following.

1. Choose a confidence probability α close to 1 (e.g. 0.95 or 0.99) and construct the confidence region Z_α for the transition probabilities using formulae (14)– (15).
2. Construct the information set $\mathbf{B}(T, Z_\alpha)$ for the multistage nonrandom system for the mean value of $B(T, \xi_T)$ with known $B(0)$ and uncertain matrix $P \in \mathbf{P}$. For the information set construction we use results obtained by A.B. Kurzhanski, M.I. Gusev, E.K. Kostousova [2, 4, 5].
3. Calculate the lower estimate of the left α-quantile of the expected portfolio profitability $q_\alpha = \min\{b | b \in \mathbf{B}(T, Z_\alpha)\}$.

Using this approach and parallelotop confidence regions we obtain for a considered portfolio $\overline{NPV} > 0.023$ with probability 0.95. If we use ellipsoid confidence regions, we get $\overline{NPV} > 0.032$ with the same probability. The result depends on a shape of a confidence region. In both cases we obtain only a lower estimate of the quantile.

Simulation method

Another approach to quantile estimation problem is a simulation method, which includes in following steps.

1. On the base of the statistical data n_{ij} about changing credits quality find the transition probabilities estimates w_{ij}.
2. Generate N times $k(k-1)$ Gaussian random vector $Z^{(n)} = \{z_{ij}^{(n)}, i, j = \overline{1,k}, j \neq i\}$ with moments $Ez_{ij} = w_{ij}$, $Cov(z_{ij} z_{ms}) = 0,\ Cov(z_{ij} z_{is}) = -{n_i}^{-1} w_{ij} w_{is},\ \ Var(z_{ij}) = {n_i}^{-1} w_{ij}(1 - w_{ij})$.
3. For generated values of the transition probabilities calculate values of the $c^{(n)}(T) = \overline{NPV}(T; P^{(n)})$ using the linear multistage moments equations.
4. Analyze the statistical distribution, moments and quantilies of $\{c^{(n)}(T) | n = \overline{1,N}\}$.

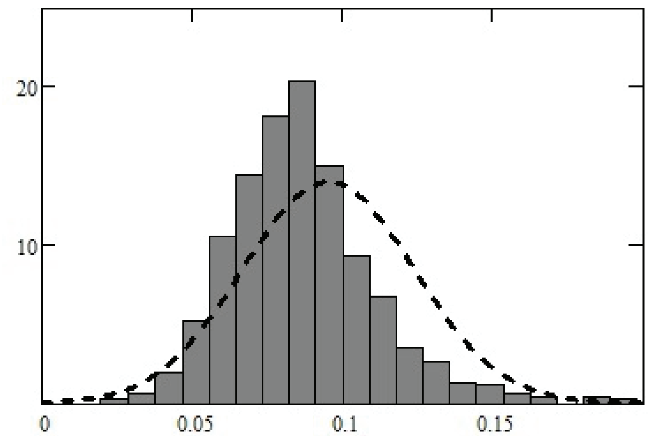

FIGURE 2. Histogram of profitability possible values

A histogram of the portfolio profitability for generated transition probability one can see in Fig. 2. The histogram is the empirical distribution of the portfolio profitability, the dashed line is the normal distribution density with the same statistical moments. We obtain using simulation method for the same portfolio $\overline{NPV}(T) > 0.043$ with probability 0.95.

The confidence method is more time-consuming because the number of estimated elements of the matrix is large and the moments equations multiplicatively depended on P. But the confidence method is not more precise, therefore the simulation method should be recommended for estimation of expected portfolio profitability.

CONCLUSION

We studied the loan portfolio dynamics described by the discrete Markov chain. It is supposed that transition probabilities are unknown and estimated during the process. By portfolio profitability we mean the expected Net Present Value. The stochastic linear equations for the portfolio profitability are obtained. Two methods to estimate the portfolio profitability are investigated. First method is based on confidence estimates, the second one is the simulation method. Obtained results apply to forecast the expected profitability of the loan portfolio.

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