ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2015, Vol. 288, Suppl. 1, pp. S1–S11.

published in Trudy Instituta Matematiki i Mekhaniki UrO RAN, 2014, Vol. 20, No. 1.

Lower Estimates for the Error of Approximation of Derivatives for Composite Finite Elements with Smoothness Property

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Received April 30, 2013

Abstract—We consider a natural class of composite finite elements that provide the mth-order smoothness of the resulting piecewise polynomial function on a triangulated domain and do not require any information on neighboring elements. It is known that, to provide a required convergence rate in the finite element method, the "smallest angle condition" must be often imposed on the triangulation of the initial domain; i.e., the smallest possible values of the smallest angles of the triangles must be lower bounded. On the other hand, the negative role of the smallest angle can be weakened (but not eliminated completely) by choosing appropriate interpolation conditions. As shown earlier, for a large number of methods of choosing interpolation conditions in the construction of simple (noncomposite) finite elements, including traditional conditions, the influence of the smallest angle of the triangle on the error of approximation of derivatives of a function by derivatives of the interpolation polynomial is essential for a number of derivatives of order 2 and higher for $m \geq 1$. In the present paper, a similar result is proved for some class of composite finite elements.
Keywords: multidimensional interpolation, finite element method, smallest angle condition, splines on triangulations.

DOI:

INTRODUCTION

Let Ω be a domain in the plane $\mathbb{R}^2$, and let $W^{n+1}M$ be the set of functions continuous on Ω together with all their partial derivatives up to order $n+1$ such that all their derivatives of order $n+1$ are bounded in absolute value by the constant M. Let a set of triangles $\Delta = \{T_1, T_2, \dots, T_N\}$ be a triangulation of the domain Ω; i.e., $\Omega = \bigcup_{i=1}^{N} T_i$ and any two triangles T_i and T_j either have no common points or have a common vertex or a common side. Two triangles sharing a side are called neighboring triangles.

Consider an arbitrary triangle $T = \langle a_1, a_2, a_3 \rangle$ from the triangulation Δ. Let T be a composite finite element; i.e., let T be divided into k triangles $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k$. We assume that this partition of T satisfies the following property: for any side $[a_i, a_j]$ $(i, j \in \{1,2,3\},\ i \neq j)$, there exists a triangle $\mathcal{T}_s$ $(1 \leq s \leq k)$ with a side coinciding with $[a_i, a_j]$. Let, on each of the triangles $\mathcal{T}_i$, a

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polynomial $P_{n,i} = P_i$ of total degree at most n be given (i.e., the sum of degrees of each monomial is at most n). Thus, a piecewise polynomial function S_T is given on T.

We will impose the following requirements on S_T.

C1. S_T interpolates values of a function f and, possibly, of its derivatives along specified directions at some points of the triangle T, including the vertices and some points on the sides, and is completely defined by interpolation conditions and the condition of belonging to the class $C^r(T)$ $(r \geq 1)$, i.e., is given on T locally.

C2. The family of all S_T forms a function from the class $C^r(\Omega)$ $(r \geq 1)$; i.e., if

$$\widetilde{S} = \sum_{T\in\Delta} \widetilde{S}_T, \qquad \text{where} \quad \widetilde{S}_T(u) = \begin{cases} S_T(u) & \text{for} \quad u \in T, \\ 0 & \text{for} \quad u \in \Omega \setminus T, \end{cases}$$

then $\widetilde{S} \in C^r(\Omega)$.

We adopt the following convention: for any ψ_1 and ψ_2 (they can be functions of some variables or constants), we write $\psi_1 \overset{(\gtrsim)}{\lesssim} \psi_2$ if there exists a number $C > 0$ independent of the function f and geometric characteristics of the triangle (we admit a dependence on k and n) such that $\psi_1 \overset{(\geq)}{\leq} C\psi_2$.

It is known that, for a simple (noncomposite) finite element $T^* \subset \mathbb{R}^m$ (T^* is not necessarily a triangle or an m-simplex) under sufficiently general constraints on the body T^* and on conditions of interpolation of a function $f \in W^{n+1}M$, there are upper estimates for the value of approximation of the function and its derivatives [1], which, for the case of a triangle, take the form

$$||D^s f - D^s P_n||_{C(T^*)} \lesssim MH^{n+1-s} (\sin\alpha)^{-s}, \quad 0 \leq s \leq n, \tag{0.1}$$

where P_n is an interpolation polynomial of Lagrange, Hermite, or Birkhoff type of total degree at most n and α is the smallest angle of the triangle (see also [2–4]). Estimates of type (0.1) are the reason for introducing a constraint on the triangulation, i.e., for imposing the “smallest angle condition,” which is the requirement on values of the smallest angles of the triangles to be separated from zero. There have been successful attempts to weaken the negative influence of the smallest angle due to an appropriate choice of interpolation conditions (see [5–16]). However, analyzing these papers, we can observe that, in estimates for derivatives of the second and higher orders (if we consider the set of all possible directions along which the derivatives are taken), the sine of the smallest angle in their denominators is absent only in cases when the continuity of the global piecewise polynomial function is provided on Ω but not its smoothness. This observation can be proved: in [17], it was shown for a simple (noncomposite) finite element that, for a large number of methods of choosing local interpolation conditions, including traditional methods, in constructing a piecewise polynomial function of global smoothness 1 or higher, the negative effect of the smallest angle of the triangle on the error of approximation of derivatives of a function by derivatives of the interpolation polynomial is essential for derivatives of order 2 and higher. In the present paper, we prove a similar result for some natural subclass of composite finite elements described above, which provide smoothness of the spline $\widetilde{S}$ on Ω without information on neighboring finite elements.

1. FORMULATION OF THE THEOREM AND BASIC NOTATION

Let $T = \langle a_1, a_2, a_3\rangle$ be a triangular composite finite element divided into triangles $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k$, and let S_T be a spline satisfying conditions C1 and C2. Denote by α, β, and θ the values of the angles of T at the vertices a_1, a_2, and a_3, respectively. We assume that $0 < \alpha \leq \beta \leq \theta$. We will also use the following notation: H is the diameter of the triangle T; ς_{ij} are the unit vectors directed from a_i to a_j; n_{ij} $(i,j = 1,2,3;\ i \neq j)$ are the unit normal vectors to the sides $[a_i,a_j]$; d_{ij} is the length of the side $[a_i,a_j]$; and $D^s_{\xi_1 \ldots \xi_s}$ are derivatives of order s along arbitrary unit vectors $\xi_1,\ldots,\xi_s$. In what follows, the norm is understood as the L_∞-norm.

Place T in a rectangular coordinate system Oxy (see figure) so that the vertices of T have the following coordinates: $a_1 = (0,0)$, $a_2 = (b+a,0)$, and $a_3 = (b,h)$, where $a,b,h > 0$, $a < b$, and $h < b$ (the last two inequalities follow from our convention about relations between values of the angles at the vertices a_1, a_2, and a_3). Obviously, $a + b = H$.

Let the restrictions of the spline $S_T(u)$ and its derivative $\partial S_T(u)/\partial n_{ij}$ to every side $[a_i,a_j]$ of the triangle be uniquely defined by interpolation conditions given at points of the side $[a_i,a_j]$. Note that, by conditions C1 and C2, the spline $S_{T^*}(u)$ on a triangle T^* neighboring T must satisfy the same conditions. In addition, assume that interpolation conditions at points of the sides of the triangle T are specified so that the following equality holds for every side $[a_i,a_j]$ of T:

$$\left.\frac{\partial^s \left(f(u) - S_T(u)\right)}{\partial n^s_{ij}}\right|_{u\in[a_i,a_j]} = \frac{1}{(n+1-s)!}\,\frac{\partial^{n+1} f(\vartheta^s_{ij})}{\partial n^s_{ij}\partial \varsigma^{n+1-s}_{ij}}\, d^{n+1-s}_{ij}\omega_{ij,n+1-s}(t)\,, \qquad (1.1)$$

$$s = 0,\ldots,r, \quad i,j \in \{1,2,3\}, \quad i \neq j,$$

where $\vartheta^s_{ij} \in [a_i,a_j]$, $\omega_{ij,n+1-s}(t)$ is a polynomial of degree $n+1-s$ with leading coefficient equal to 1, and $t = |u - a_i|/d_{ij} \in [0,1]$ ($|u - a_i|$ denotes the distance between the points u and a_i).

Equality (1.1) is a formula for the remainder term in the interpolation formula for the function $\partial^s f/\partial n^s_{ij}$ and its interpolation polynomial given on the line segment $[a_i,a_j]$, which is the restriction of the spline S_T to this segment. Since, to prove the theorem, it is sufficient that the function $\widetilde{S}$ belong to the class $C^1(\Omega)$, we can assume without loss of generality that $r = 1$.

Recall that conditions for constructing the spline S_T on the triangle T include the requirement of smoothness of the resulting spline $\widetilde{S}$ on Ω with no information about finite elements neighboring T. Usually, in this case, interpolation conditions of the same type are specified for all sides of the triangle T; i.e.,

$$\omega_{ij,n+1-s}(x) = \omega_{pq,n+1-s}(x)$$

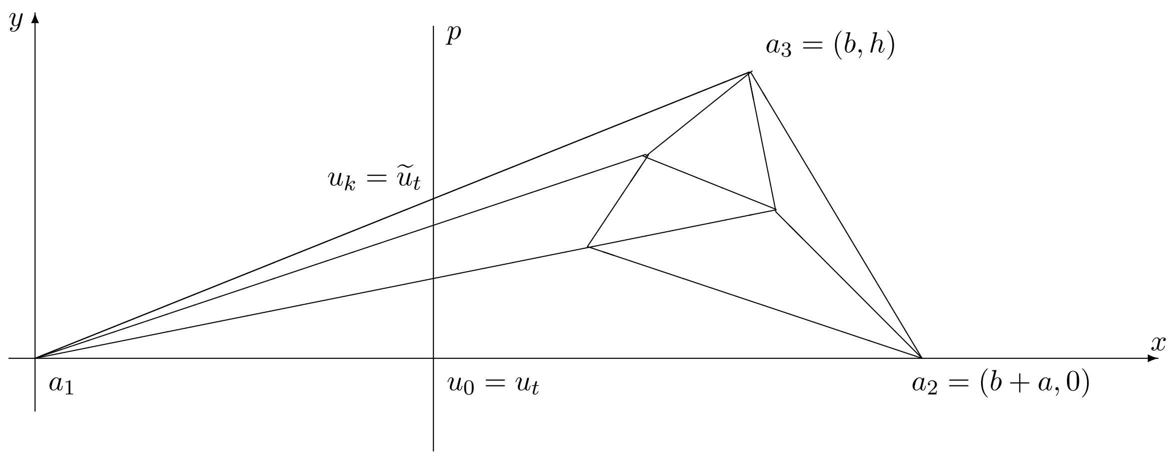

Figure. A composite element T.

for any i, j, p, and q and $s = 0, 1$ (or $s = 0, \dots, r$ if $r > 1$). This means that

$$\omega_{ij,n+1-s}(x) = (-1)^{n+1-s}\omega_{ij,n+1-s}(1-x).$$

Then,

$$\omega^{(n-s)}_{ij,n+1-s}(x) = -\omega^{(n-s)}_{ij,n+1-s}(1-x). \tag{1.2}$$

Since $\omega_{ij,n+1-s}$ is a polynomial of degree $n+1-s$ with leading coefficient 1, we see that $\omega^{(n-s)}_{ij,n+1-s}(x)$ is a linear function and relation (1.2) implies that

$$\frac{\omega^{(n-s)}_{ij,n+1-s}(x)}{(n+1-s)!} = x - \frac{1}{2} \tag{1.3}$$

for any $s = 0, 1$ and $i, j \in \{1, 2, 3\}$, $i \neq j$.

Consider the function

$$f^*(x,y) = \delta_1 M \frac{x^{n+1}}{(n+1)!} + \delta_2 M \frac{x^n y}{n!}, \tag{1.4}$$

where δ_1 and δ_2 are such that $f^* \in W^{n+1}M$ and $\delta_1^2 + \delta_2^2 \neq 0$ (a similar function was used in [8] for $n = 5$ to prove that the estimates for the error of approximation of derivatives obtained there are unimprovable). We set

$$e(x,y) = f^*(x,y) - S_T(x,y),$$
$$e_i(x,y) = (f^*(x,y) - S_T(x,y))\big|_{\mathcal{T}_i} = f^*(x,y) - P_{n,i}(x,y).$$

Theorem. *If conditions* $C1$ *and* $C2$ *hold for* S_T *and relations* (1.1) *and* (1.3) *are satisfied, then, for any* $s = 2, \dots, n$, *there exist* $\alpha_0 > 0$ *and unit vectors* $\xi_1, \dots, \xi_s$ *such that the following estimates hold for any* $\alpha < \alpha_0$:

$$\left\| D^s_{\xi_1 \dots \xi_s}(f^* - S_T) \right\| \gtrsim \frac{MH^{n+1-s}}{\sin\alpha}. \tag{1.5}$$

2. PROOF OF THE THEOREM

Recall that there is a finite number k of triangles $\mathcal{T}_i$ in the partition of T. Let $\mathcal{T}_i = \langle a_1^{(i)}, a_2^{(i)}, a_3^{(i)} \rangle$ ($i = 1, \dots, k$). Then, there exists a nonempty intersection of a "vertical" strip of width $\widetilde{H} \gtrsim H$ and the triangle T such that any straight line lying in this strip (and, hence, parallel to the y-axis) intersects only those sides $[a_r^{(i)}, a_s^{(i)}]$ ($1 \le s, r \le 3$, $s \neq r$) of triangles $\mathcal{T}_i$ from the set $\{\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k\}$ for which $\left|a_s^{(i)} - a_r^{(i)}\right| \gtrsim H$. More exactly, we can consider segments $Q \subset [a_1, a_2]$ and $\widetilde{Q} \subset [a_1, a_3]$ on the sides $[a_1, a_2]$ and $[a_1, a_3]$ of the composite element T with the following properties:

1°. $|Q| = c(k)H$, where $|Q|$ is the length of the segment Q and $c(k)$ is a positive number depending only on k (i.e., $|Q| \gtrsim H$).

2°. Let straight lines p_1 and p_2 be parallel to the y-axis and pass through two different points $q_1, q_2 \in Q$. Let p_1 intersect the side $[a_r^{(i)}, a_s^{(i)}]$ of some triangle $\mathcal{T}_i$ at some point u_1. Then, p_2 also intersects $[a_r^{(i)}, a_s^{(i)}]$ at some point u_2.

3°. Every straight line p parallel to the y-axis and passing through any point $u \in Q$ intersects the segment $\widetilde{Q}$ at some point $\widetilde{u}$ (obviously, $\left|\widetilde{Q}\right| \gtrsim H$).

The segments Q and $\widetilde{Q}$ can be represented as follows:

$$Q = \{u = a_1 + t\,(a_2 - a_1) : t \in \sigma \subseteq [0,1]\}, \qquad \widetilde{Q} = \left\{\widetilde{u} = a_1 + \widetilde{t}\,(a_3 - a_1) : \widetilde{t} \in \widetilde{\sigma} \subseteq [0,1]\right\},$$

where $|\sigma| \gtrsim 1$ and $|\widetilde{\sigma}| \gtrsim 1$.

It is sufficient to consider one straight line from this strip. Take some value $t \in \sigma$ and the corresponding point $u_t = a_1 + t\,(a_2 - a_1)$. Draw a straight line p parallel to the y-axis through the point u_t. This line intersects the segment $\widetilde{Q}$ at some point $\widetilde{u}_t = a_1 + \widetilde{t}\,(a_3 - a_1)$ for the corresponding $\widetilde{t}$. Thus, we obtain points u_t and $\widetilde{u}_t$ and a function $\psi : \sigma \to \widetilde{\sigma}$ such that $\widetilde{t} = \psi(t)$.

Consider all triangles of the composite finite element T having common points with the line p (without loss of generality, we can assume that p has empty intersection with the set of all vertices of triangles that make up the triangulation of T; we can also assume that the number of the triangles is k) and enumerate them as follows: $\mathcal{T}_1$ is the triangle in which one of the sides intersected by the line p coincides with $[a_1, a_2]$, $\mathcal{T}_2$ is the triangle neighboring $\mathcal{T}_1$, $\ldots$, and $\mathcal{T}_k$ is the triangle neighboring $\mathcal{T}_{k-1}$.

We use the following notation for the sides of these triangles: $[c_1^0, c_2^0]$ is the side of the triangle $\mathcal{T}_1$ coinciding with $[a_1, a_2]$, $[c_1^1, c_2^1]$ is the common side of the triangles $\mathcal{T}_1$ and $\mathcal{T}_2$, $\ldots$, $[c_1^{k-1}, c_2^{k-1}]$ is the common side of the triangles $\mathcal{T}_{k-1}$ and $\mathcal{T}_k$, and $[c_1^k, c_2^k]$ is the side of the triangle $\mathcal{T}_k$ coinciding with $[a_1, a_3]$. By conditions 1°–3°, we can assert that the inequality

$$\left|c_2^j - c_1^j\right| \gtrsim H \tag{2.1}$$

holds for every $j = 1, \ldots, k$. Further, let u_j be the intersection points of the line p with the segments $[c_1^j, c_2^j]$ $(j = 0, \ldots, k)$. In particular, $u_0 = u_t$ and $u_k = \widetilde{u}_t$.

Let (x_i^j, y_i^j) be the coordinates of the point c_i^j, and let the inequalities $x_1^j < x_2^j$ hold for all j. Denote by τ_j the unit vectors directed from c_1^j to c_2^j. Obviously, $c_1^{j-1} = c_1^j$ or $c_2^{j-1} = c_2^j$ (in what follows, it does not matter), $\tau_0 = \varsigma_{12}$, and $\tau_k = \varsigma_{13}$.

Denote by α_j $(j = 1, \ldots, k)$ the angles between the vectors τ_{j-1} and τ_j with regard to the directions of these vectors: if, after drawing them from the same point, the minimum angle of rotation from τ_{j-1} to τ_j is counterclockwise, then $\alpha_j > 0$; otherwise, $\alpha_j < 0$. Note that

$$\sum_{j=1}^{k} \alpha_j = \alpha. \tag{2.2}$$

Consider the functions $\omega_{ij,n+1-k}$ from (1.1).

Lemma 1. *Assume that $\widetilde{t} = \psi(t)$ and the condition $a < b/(2n)$ holds. Define*

$$W_2(t) = \frac{(n+1)d_{13}\ \omega_{13,n+1}^{(n)}\left(\ \widetilde{t}\ \right)\cos\alpha}{(n+1)!} - \frac{n d_{12}\ \omega_{12,n}^{(n-1)}(t)}{n!} - \frac{|u_k - u_0|}{\tan\alpha}\ .$$

Then,

$$|W_2(t)| \gtrsim H. \tag{2.3}$$

Proof. Since the line p passing through the points u_0 and u_k is parallel to the y-axis, $|u_k - a_1| = d_{13}\widetilde{t}$, and $|u_0 - a_1| = d_{12}t$, we have

$$d_{13}\widetilde{t}\cos\alpha = d_{12}t,$$

which yields

$$\widetilde{t} = \frac{d_{12}}{d_{13}}\ \frac{t}{\cos\alpha} = \frac{b+a}{(b^2+h^2)^{1/2}}\frac{(b^2+h^2)^{1/2}}{b}\ t = \frac{b+a}{b}\ t.$$

Note that

$$\frac{|u_k - u_0|}{\tan\alpha} = |u_0 - a_1| = (b+a)t.$$

Then, in view of (1.3) and the relations $d_{12} = b+a$ and $d_{13}\cos\alpha = \left(b^2+h^2\right)^{1/2}\cos\alpha = b$, we obtain

$$W_2(t) = (n+1)b\Big(\widetilde{t} - \frac{1}{2}\Big) - n(b+a)\Big(t - \frac{1}{2}\Big) - (b+a)t$$

$$= (n+1)b\Big(\frac{b+a}{b}\ t - \frac{1}{2}\Big) - n(b+a)\Big(t - \frac{1}{2}\Big) - (b+a)t = -\frac{b}{2} + \frac{na}{2}.$$

Thus,

$$|W_2(t)| \geq \frac{b}{2} - \frac{nb}{4n} \geq \frac{b}{4} \gtrsim H. \qquad \square$$

Lemma 2. *Assume that $\widetilde{t} = \psi(t)$ and the condition $a \geq b/(2n)$ holds. Define*

$$W_1(t) = \frac{d_{13}\ \omega^{(n)}_{13,n+1}\left(\ \widetilde{t}\ \right)\cos\alpha}{(n+1)!} - \frac{d_{12}\ \omega^{(n)}_{12,n+1}\left(\ t\ \right)}{(n+1)!}.$$

Then,

$$|W_1(t)| \gtrsim H. \tag{2.4}$$

Proof. The proof is similar to the proof of Lemma 1:

$$W_1(t) = b\Big(\widetilde{t} - \frac{1}{2}\Big) - (b+a)\Big(t - \frac{1}{2}\Big) = b\Big(\frac{b+a}{b}\ t - \frac{1}{2}\Big) - (b+a)\Big(t - \frac{1}{2}\Big) = \frac{a}{2}\ ,$$

which implies the estimate $|W_1(t)| \geq b/(4n) \gtrsim H$. $\square$

Lemma 3. *Let $\tau = (\cos\varphi, \sin\varphi)$. Then, the following equality holds for any function g*:

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial \tau}\ \frac{1}{\cos\varphi} - \frac{\partial g}{\partial y}\tan\varphi. \tag{2.5}$$

Proof. The proof follows from the relation $\partial g/\partial\tau = (\partial g/\partial x)\cos\varphi + (\partial g/\partial x)\sin\varphi$. $\square$

Let us introduce the notation

$$\gamma_j = \sum_{s=1}^{j} \alpha_s, \tag{2.6}$$

where $j = 1, \dots, k$, and set $\gamma_0 = 0$.

Lemma 4. *The following equality holds*:

$$\sum_{s=0}^{k-1} \frac{\sin\alpha_{k-s}}{\cos\gamma_{k-s}\cos\gamma_{k-s-1}} = \tan\alpha. \tag{2.7}$$

Proof. Consider the sum of the last two terms:

$$\frac{\sin\alpha_2}{\cos\gamma_2\cos\gamma_1} + \frac{\sin\alpha_1}{\cos\gamma_1} = \frac{\sin\alpha_2 + \sin\alpha_1\cos(\alpha_1+\alpha_2)}{\cos(\alpha_1+\alpha_2)\cos\alpha_1} = \frac{\sin\alpha_2 + \left(-\sin\alpha_2 + \sin(2\alpha_1+\alpha_2)\right)/2}{\cos(\alpha_1+\alpha_2)\cos\alpha_1}$$

$$= \frac{\sin\alpha_2 + \sin(2\alpha_1+\alpha_2)}{2\cos(\alpha_1+\alpha_2)\cos\alpha_1} = \frac{2\sin(\alpha_1+\alpha_2)\cos\alpha_1}{2\cos(\alpha_1+\alpha_2)\cos\alpha_1} = \frac{\sin(\alpha_1+\alpha_2)}{\cos(\alpha_1+\alpha_2)}\ . \tag{2.8}$$

Then, setting $\alpha^* = \alpha_1 + \alpha_2$ and using (2.8), we can represent the sum of the last three terms as

$$\frac{\sin\alpha_3}{\cos\gamma_3\cos\gamma_2} + \frac{\sin\alpha_2}{\cos\gamma_2\cos\gamma_1} + \frac{\sin\alpha_1}{\cos\gamma_1} = \frac{\sin\alpha_3}{\cos\gamma_3\cos\gamma_2} + \frac{\sin(\alpha_1+\alpha_2)}{\cos(\alpha_1+\alpha_2)}$$

$$= \frac{\sin\alpha_3}{\cos(\alpha^*+\alpha_3)\cos\alpha^*} + \frac{\sin\alpha^*}{\cos\alpha^*} = \frac{\sin(\alpha^*+\alpha_3)}{\cos(\alpha^*+\alpha_3)} = \frac{\sin(\alpha_1+\alpha_2+\alpha_3)}{\cos(\alpha_1+\alpha_2+\alpha_3)}\ .$$

Further, acting by induction, we obtain (2.7). □

Lemma 5. *For any function g defined on the triangle T and numbers $i = 1,\ldots,k$ and $s = n-1, n$, the following representation holds with $|C_{i,j}| \lesssim 1$:*

$$\frac{\partial^s g}{\partial\tau_i^s} = \frac{\partial^s g}{\partial\tau_{i-1}^s}\ \frac{\cos^s\gamma_i}{\cos^s\gamma_{i-1}} + \frac{\partial^s g}{\partial\tau_{i-1}^{s-1}\partial y}\ \frac{s\cos^{s-1}\gamma_i\sin\alpha_i}{\cos^s\gamma_{i-1}} + \sum_{j=2}^{s} C_{i,j}\frac{\partial^s g}{\partial\tau_{i-1}^{s-j}\partial y^j}\ \sin^j\alpha_i. \tag{2.9}$$

Proof. Since $\tau_i = (\cos\gamma_i, \sin\gamma_i)$ (see (2.6)), taking into account (2.5), we obtain

$$\frac{\partial^s g}{\partial\tau_i^s} = \Big(\frac{\partial}{\partial x}\cos\gamma_i + \frac{\partial}{\partial y}\sin\gamma_i\Big)^s g = \Big(\Big(\frac{\partial}{\partial\tau_{i-1}}\ \frac{1}{\cos\gamma_{i-1}} - \frac{\partial}{\partial y}\ \frac{\sin\gamma_{i-1}}{\cos\gamma_{i-1}}\Big)\cos\gamma_i + \frac{\partial}{\partial y}\sin\gamma_i\Big)^s g$$

$$= \Big(\frac{\partial}{\partial\tau_{i-1}}\ \frac{\cos\gamma_i}{\cos\gamma_{i-1}} + \frac{\partial}{\partial y}\Big(\sin\gamma_i - \frac{\cos\gamma_i}{\cos\gamma_{i-1}}\ \sin\gamma_{i-1}\Big)\Big)^s g.$$

Since

$$\sin\gamma_i - \frac{\cos\gamma_i}{\cos\gamma_{i-1}}\ \sin\gamma_{i-1} = \sin(\gamma_{i-1}+\alpha_i) - \frac{\cos(\gamma_{i-1}+\alpha_i)}{\cos\gamma_{i-1}}\ \sin\gamma_{i-1}$$

$$= \sin\gamma_{i-1}\cos\alpha_i + \cos\gamma_{i-1}\sin\alpha_i - \frac{\cos\gamma_{i-1}\cos\alpha_i}{\cos\gamma_{i-1}}\ \sin\gamma_{i-1} + \frac{\sin\gamma_{i-1}\sin\alpha_i}{\cos\gamma_{i-1}}\ \sin\gamma_{i-1}$$

$$= \cos\gamma_{i-1}\sin\alpha_i + \tan\gamma_{i-1}\sin\gamma_{i-1}\sin\alpha_i = \sin\alpha_i\Big(\cos\gamma_{i-1} + \frac{\sin^2\gamma_{i-1}}{\cos\gamma_{i-1}}\Big) = \frac{\sin\alpha_i}{\cos\gamma_{i-1}}\ ,$$

we have

$$\frac{\partial^s g}{\partial\tau_i^s} = \Big(\frac{\partial}{\partial\tau_{i-1}}\frac{\cos\gamma_i}{\cos\gamma_{i-1}} + \frac{\partial}{\partial y}\frac{\sin\alpha_i}{\cos\gamma_{i-1}}\Big)^s g.$$

Removing the parentheses, we obtain (2.9). □

Let h_i be the altitude of the triangle $\mathcal{T}_i$ drawn to the side $[c_1^s, c_2^s]$ ($s = i-1$ or i). Since $\mu(\mathcal{T}_i) < \mu(T)$ (where μ denotes the area of the corresponding triangle) and $|c_2^s - c_1^s| \gtrsim H$, we can assert that $h_i \lesssim h$. In particular,

$$|\sin\alpha_i| \lesssim \sin\alpha, \qquad i = 1,\ldots,k. \tag{2.10}$$

Lemma 6. *The following expansion is valid*:

$$\frac{\partial^n e_k(u_k)}{\partial\tau_k^n} = \frac{\partial^n e_1(u_0)}{\partial x^n}\ \cos^n\alpha + \frac{\partial^n e_1(u_0)}{\partial x^{n-1}\partial y}\ n\cos^n\alpha\tan\alpha$$

$$+\sum_{r=1}^{k}\sum_{j=2}^{n} D_{r,j}\frac{\partial^n e_r(u_r)}{\partial\tau_{r-1}^{n-j}\partial y^j}\ \sin^{j-1}\alpha_r\sin\alpha+\delta_2 M\,|u_k-u_0|\cos^n\alpha, \tag{2.11}$$

where the quantities $D_{r,j}$ satisfy the inequality $|D_{r,j}|\lesssim 1$.

Proof. Setting $g=e_k$, $i=k$, and $s=n$ in (2.9), we obtain the equality

$$\frac{\partial^n e_k(u_k)}{\partial\tau_k^n}=\frac{\partial^n e_k(u_k)}{\partial\tau_{k-1}^n}\ \frac{\cos^n\gamma_k}{\cos^n\gamma_{k-1}}+n\,\frac{\partial^n e_k(u_k)}{\partial\tau_{k-1}^{n-1}\partial y}\ \frac{\cos^{n-1}\gamma_k\sin\alpha_k}{\cos^n\gamma_{k-1}}+\sum_{j=2}^{n}D_{k,j}\frac{\partial^n e_k(u_k)}{\partial\tau_{k-1}^{n-j}\partial y^j}\ \sin^j\alpha_k.$$

Let us expand $\partial^n e_k(u_k)/\partial\tau_{k-1}^n$ and $\partial^n e_k(u_k)/\left(\partial\tau_{k-1}^{n-1}\partial y\right)$ by the Lagrange formula of finite increments at the point u_{k-1} (recall that the line p passing through u_k and u_{k-1} is parallel to the y-axis). Then,

$$\frac{\partial^n e_k(u_k)}{\partial\tau_k^n}=\frac{\partial^n e_k(u_{k-1})}{\partial\tau_{k-1}^n}\ \frac{\cos^n\gamma_k}{\cos^n\gamma_{k-1}}+n\,\frac{\partial^n e_k(u_{k-1})}{\partial\tau_{k-1}^{n-1}\partial y}\ \frac{\cos^{n-1}\gamma_k\sin\alpha_k}{\cos^n\gamma_{k-1}}$$

$$+\sum_{j=2}^{n}D_{k,j}\frac{\partial^n e_k(u_k)}{\partial\tau_{k-1}^{n-j}\partial y^j}\ \sin^j\alpha_k+\frac{\partial^{n+1}f^*}{\partial\tau_{k-1}^n\partial y}\,|u_k-u_{k-1}|\ \frac{\cos^n\gamma_k}{\cos^n\gamma_{k-1}}$$

$$+\,n\frac{\partial^{n+1}f^*}{\partial\tau_{k-1}^{n-1}\partial y^2}\,|u_k-u_{k-1}|\ \frac{\cos^{n-1}\gamma_k\sin\alpha_k}{\cos^n\gamma_{k-1}}\ .$$

Since f^* has form (1.4), the last term is zero. Further, since $\tau_{k-1}=(\cos\gamma_{k-1},\sin\gamma_{k-1})$, formula (1.4) implies the equality

$$\frac{\partial^{n+1}f^*}{\partial\tau_{k-1}^n\partial y}=\delta_2 M\cos^n\gamma_{k-1}.$$

In view of the smoothness of the function $e(x,y)$ on the triangle T, we obtain

$$\frac{\partial^n e_k(u_k)}{\partial\tau_k^n}=\frac{\partial^n e_{k-1}(u_{k-1})}{\partial\tau_{k-1}^n}\ \frac{\cos^n\gamma_k}{\cos^n\gamma_{k-1}}+n\,\frac{\partial^n e_{k-1}(u_{k-1})}{\partial\tau_{k-1}^{n-1}\partial y}\ \frac{\cos^{n-1}\gamma_k\sin\alpha_k}{\cos^n\gamma_{k-1}}$$

$$+\sum_{j=2}^{n}D_{k,j}\frac{\partial^n e_k(u_k)}{\partial\tau_{k-1}^{n-j}\partial y^j}\ \sin^j\alpha_k+\delta_2 M\,|u_k-u_{k-1}|\cos^n\gamma_k.$$

Taking into account (2.10), we apply (2.9) to the cases $g=e_{k-1}$, $i=k-1$, and $s=n$ and $g=\partial e_k/\partial y$, $i=k-1$, and $s=n-1$:

$$\frac{\partial^n e_k(u_k)}{\partial\tau_k^n}=\frac{\partial^n e_{k-1}(u_{k-1})}{\partial\tau_{k-2}^n}\ \frac{\cos^n\gamma_k}{\cos^n\gamma_{k-2}}$$

$$+\,\frac{\partial^n e_{k-1}(u_{k-1})}{\partial\tau_{k-2}^{n-1}\partial y}\ \frac{n\cos^n\gamma_k}{\cos^{n-1}\gamma_{k-2}}\left(\frac{\sin\alpha_{k-1}}{\cos\gamma_{k-1}\cos\gamma_{k-2}}+\frac{\sin\alpha_k}{\cos\gamma_k\cos\gamma_{k-1}}\right)$$

$$+\sum_{r=k-1}^{k}\sum_{j=2}^{n}D_{r,j}\frac{\partial^n e_r(u_r)}{\partial\tau_{r-1}^{n-j}\partial y^j}\ \sin^{j-1}\alpha_r\sin\alpha+\delta_2 M\,|u_k-u_{k-1}|\cos^n\gamma_k.$$

As earlier, we expand $\partial^n e_{k-1}(u_{k-1})/\partial\tau_{k-2}^n$ and $\partial^n e_{k-1}(u_{k-1})/\big(\partial\tau_{k-2}^{n-1}\partial y\big)$ by the Lagrange formula of finite increments at the point u_{k-2} and use the smoothness of the function $e(x,y)$ and the fact that $\partial^{n+1}f^*/\big(\partial\tau_{k-2}^n\partial y\big)=\delta_2 M\cos^n\gamma_{k-2}$. We obtain

$$\frac{\partial^n e_k(u_k)}{\partial\tau_k^n}=\frac{\partial^n e_{k-2}(u_{k-2})}{\partial\tau_{k-2}^n}\;\frac{\cos^n\gamma_k}{\cos^n\gamma_{k-2}}$$

$$+\frac{\partial^n e_{k-2}(u_{k-2})}{\partial\tau_{k-2}^{n-1}\partial y}\;\frac{n\cos^n\gamma_k}{\cos^{n-1}\gamma_{k-2}}\Big(\frac{\sin\alpha_{k-1}}{\cos\gamma_{k-1}\cos\gamma_{k-2}}+\frac{\sin\alpha_k}{\cos\gamma_k\cos\gamma_{k-1}}\Big)$$

$$+\sum_{r=k-1}^{k}\sum_{j=2}^{n}D_{r,j}\frac{\partial^n e_r(u_r)}{\partial\tau_{r-1}^{n-j}\partial y^j}\;\sin^{j-1}\alpha_r\sin\alpha+\delta_2 M\left(|u_k-u_{k-1}|+|u_{k-1}-u_{k-2}|\right)\cos^n\gamma_k.$$

Continuing this process, we come to the equality

$$\frac{\partial^n e_k(u_k)}{\partial\tau_k^n}=\frac{\partial^n e_1(u_0)}{\partial\tau_0^n}\;\frac{\cos^n\gamma_k}{\cos^n\gamma_0}+\frac{\partial^n e_1(u_0)}{\partial\tau_0^{n-1}\partial y}\;\frac{n\cos^n\gamma_k}{\cos^{n-1}\gamma_0}\sum_{s=0}^{k-1}\frac{\sin\alpha_{k-s}}{\cos\gamma_{k-s}\cos\gamma_{k-s-1}}$$

$$+\sum_{r=1}^{k}\sum_{j=2}^{n}D_{r,j}\frac{\partial^n e_r(u_r)}{\partial\tau_{r-1}^{n-j}\partial y^j}\;\sin^{j-1}\alpha_r\sin\alpha+\delta_2 M\cos^n\gamma_k\sum_{i=1}^{k}|u_i-u_{i-1}|\,.$$

Taking into account (2.7) and the relations $\sum_{i=1}^k|u_i-u_{i-1}|=|u_k-u_0|\,,\ \partial/\partial\tau_0=\partial/\partial x,\ \gamma_k=\alpha,$ and $\gamma_0=0$, we come to (2.11). ☐

Lemma 7. *There exist $r\in\{1,\dots,k\}$ and $j\in\{2,\dots,n\}$ such that*

$$\left|\frac{\partial^n e_r(u_r)}{\partial\tau_{r-1}^{n-j}\partial y^j}\right|\gtrsim\frac{MH}{|\sin^{j-1}\alpha_r|}\;. \tag{2.12}$$

Proof. Using (1.1), the form of the function f^*, and the relation $\tau_k=\varsigma_{13}=(\cos\alpha,\sin\alpha)$, we obtain

$$\frac{\partial^n e_k(u_k)}{\partial\tau_k^n}=\frac{1}{(n+1)!}\;\frac{\partial^{n+1}f^*}{\partial\tau_k^{n+1}}\;d_{13}\omega_{13,n+1}^{(n)}\left(\widetilde{t}\,\right)$$

$$=\frac{1}{(n+1)!}\left(\delta_1 M\cos^{n+1}\alpha+\delta_2(n+1)M\cos^n\alpha\sin\alpha\right)d_{13}\omega_{13,n+1}^{(n)}\left(\widetilde{t}\,\right);$$

$$\frac{\partial^n e_1(u_0)}{\partial x^n}=\frac{1}{(n+1)!}\;\frac{\partial^{n+1}f^*}{\partial x^{n+1}}\;d_{12}\omega_{12,n+1}^{(n)}(t)=\frac{1}{(n+1)!}\delta_1 M d_{12}\omega_{12,n+1}(t);$$

$$\frac{\partial^n e_1(u_0)}{\partial x^{n-1}\partial y}\;=\frac{1}{n!}\;\frac{\partial^{n+1}f^*}{\partial x^n\partial y}d_{12}\omega_{12,n}^{(n-1)}=\frac{1}{n!}\;\delta_2 M d_{12}\omega_{12,n}^{(n-1)}.$$

Then, (2.11) can be rewritten as

$$\sum_{r=1}^{k}\sum_{j=2}^{n}D_{r,j}\frac{\partial^n e_r(u_r)}{\partial\tau_{r-1}^{n-j}\partial y^j}\;\sin^{j-1}\alpha_r\sin\alpha=\delta_1 M\cos^n\alpha\Bigg(\frac{d_{13}\;\omega_{13,n+1}^{(n)}\left(\,\widetilde{t}\,\right)\cos\alpha}{(n+1)!}-\frac{d_{12}\;\omega_{12,n+1}^{(n)}\left(\,t\,\right)}{(n+1)!}\Bigg)$$

$$+\,\delta_2 M\cos^{n-1}\alpha\sin\alpha\Bigg(\frac{(n+1)d_{13}\;\omega_{13,n+1}^{(n)}\left(\,\widetilde{t}\,\right)\cos\alpha}{(n+1)!}-\frac{nd_{12}\;\omega_{12,n}^{(n-1)}(t)}{n!}-\frac{|u_k-u_0|}{\tan\alpha}\Bigg)$$

$$= \delta_1 M W_1(t) \cos^n \alpha + \delta_2 M W_2(t) \cos^{n-1} \alpha \sin \alpha, \tag{2.13}$$

where $W_1(t)$ and $W_2(t)$ are defined in Lemmas 1 and 2.

Choosing the values of δ_1 and δ_2 so that both terms on the right-hand side of (2.13) have identical signs and applying (2.3) and (2.4), we obtain

$$\left| \sum_{r=1}^{k} \sum_{j=2}^{n} D_{r,j} \frac{\partial^n e_r(u_r)}{\partial \tau_{r-1}^{n-j} \partial y^j} \ \sin^{j-1} \alpha_r \sin \alpha \right| \gtrsim H \sin \alpha,$$

which implies (2.12). Lemma 7 is proved. □

Let us complete the proof of the theorem. Let r and j be such that (2.12) holds. Consider the triangle $\mathcal{T}_r$, and let q_0 be its center of mass. Since $j \geq 2$ and f^* has form (1.4), the value on the left-hand side of (2.12) is constant on $\mathcal{T}_r$ and, then,

$$\left| \frac{\partial^n e_r(q_0)}{\partial \tau_{r-1}^{n-j} \partial y^j} \right| \gtrsim \frac{MH}{\left|\sin^{j-1} \alpha_r\right|} \ .$$

Note also that the function $e_r = (f^* - S_T)\Big|_{\mathcal{T}_r}$ considered on $\mathcal{T}_r$ is a polynomial.

Let the line p_1 be parallel to the vector τ_{r-1} and pass through the point q_0. Consider the segment $Q_1 = p_1 \cap \mathcal{T}_r$. Since p_1 is parallel to the side $[c_1^{r-1}, c_2^{r-1}]$ of the triangle $\mathcal{T}_r$ and passes through its center of mass, we have in view of (2.1) the inequality $|Q_1| \gtrsim H$. Applying Markov's inequality [18, Sect. 3.5] $n - j$ times on the segment Q_1, we come to the existence of points $q_j, q_{j+1}, \ldots, q_{n-1} \in Q_1$ satisfying the inequalities

$$\left| \frac{\partial^{n-s} e_r(q_{n-s})}{\partial \tau_{r-1}^{n-j-s} \partial y^j} \right| \gtrsim \frac{MH^{s+1}}{\left|\sin^{j-1} \alpha_r\right|}, \qquad s = 1, \ldots, n-j. \tag{2.14}$$

Consider (2.14) for $s = n - j$. Let the line p_2 pass through the point q_j parallel to the y-axis. Consider the segment $Q_2 = p_2 \cap \mathcal{T}_r$. In view of the position of the point q_j ($q_j \in Q_1$, where Q_1 is the segment passing through the center of mass of $\mathcal{T}_r$ parallel to the side $[c_1^{r-1}, c_2^{r-1}]$, for which (2.1) holds), we can assert that

$$|Q_2| \gtrsim h_r,$$

where h_r is the smallest altitude of the triangle $\mathcal{T}_r$ (in view of the definition of the sign "$\gtrsim$" and (2.1), we can assume that h_r is the altitude drawn to $[c_1^{r-1}, c_2^{r-1}]$). Applying Markov's inequality $j - 2$ times on the segment Q_2, we come to the existence of points $q_2, q_3, \ldots, q_{j-1} \in Q_2$ such that

$$\left| \frac{\partial^s e_r(q_s)}{\partial y^s} \right| \gtrsim \frac{MH^{n+1-s}}{\left|\sin^{s-1} \alpha_r\right|}, \qquad s = 2, \ldots, j. \tag{2.15}$$

Combining (2.14), (2.15), and (2.10), we obtain (1.5). The theorem is proved. □

Remark. Condition (1.3) in the formulation of the theorem can be replaced by more illustrative condition (1.2) or more general conditions (2.3) and (2.4).

Proof. It was shown above that (1.3) is a consequence of (1.2). On the other hand, condition (1.3) was used only for the proof of (2.3) and (2.4). □

ACKNOWLEDGMENTS

This work was supported by the Program of the Division of Mathematical Sciences of the Russian Academy of Sciences "Modern Problems of Theoretical Mathematics," by the Ural Branch of the Russian Academy of Sciences (project no. 12-T-1-1003/4), by the RF President's Grant for State Support of Leading Scientific Schools (project no. NSh-4538.2014.1), by the Program of Joint Research of Scientists of the Ural and Siberian Branches of the Russian Academy of Sciences (project no. 12-S-1-1018), and by the Program for State Support of Leading Universities of the Russian Federation (agreement no. 02.A03.21.0006 of August 27, 2013).

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Translated by M. Deikalova