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On an Optimal Control Problem for a Nonlinear Economic Model

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Abstract

We consider a nonlinear control system describing one economic growth model. An optimal control problem for this system is posed in the present paper and an extremal functional describing the wealth of the region chosen depending to a regional consumption. Conditions were found in which the possible use of the Pontryagin maximum principle. By using characteristics of a specific region from the basic scenario of the integrated assessment model MERGE, we find a numerical solution of the posed optimal control problem.

Mathematics Subject Classification: 34H05, 49J15, 93C10

Keywords: integrated assessment model for evaluating greenhouse gases reduction policies, optimal control, Pontryagin maximum principle

1 Introduction

Global climate change is a complicated and controversial concept; many of the issues associated with it, continue to cause heated discussions. Researches conducted to date have not yet allowed to study the nature and characteristics of the phenomenon fully, moreover, an assessment of the impact of climate change on ecological, social and economical systems of regions seems to be quite a daunting task. So to study and predict changes in the characteristics of different regions integrated assessment models are used; one of them — MERGE, suggested in [1, 2] and modified in [3, 4]. This model provides an environment for the study of climate changes and assessment of their impact on the development of socio-economic systems and also for the construction of scenarios of development of regions. The complex nature of the model gives the integration of the various submodels, such as the climatic submodel, economic-energy submodel and damage assessment submodel.

The proposed research is closely related with the economic and energy submodel of MERGE. This submodel is designed to simulate the economic-wide indexes of the regions on a large time interval. It is a fully integrated applied model of the global economic equilibrium. In each moment of time supply and demand are equalized by means of the energy costs as well as cost of the notional generic merchandize uniting all goods produced outside the energy sector. Each region is submitted as a single producer-consumer. Investment decisions in each region are simulated by such choice of sequences of consumption levels to maximize the sum of discounted utilities of consumption. The present model is a discrete with a possible nonuniform mesh of the considering time interval. Optimal trajectories of components of the economic-energy systems of regions are found by using the intertemporal optimization and by maximizing the sum of intertemporal discounted utilities corresponding to the each region. The optimization problem in the present case is a problem of nonlinear programming and for its solution the iterative joint maximization method is used [5].

2 Preliminary Notes

We consider a regional economic growth model similar to that one used in MERGE. Dynamics of the main characteristics of the each region are described by the system obtained using classical production function of Cobb–Douglas:

$$\begin{aligned}
\dot{Y} &= -\mu Y + (au^{\rho\alpha} l^{\rho(1-\alpha)} + bv^{\rho\beta} w^{\rho(1-\beta)})^{1/\rho}, \\
\dot{K} &= -\mu K + u, \\
\dot{E} &= -\mu E + v, \\
\dot{N} &= -\mu N + w,
\end{aligned} \tag{1}$$

where $Y(t)$ is the economic output in every period t; $K(t)$ is the capital; $E(t)$ is the electricity; $N(t)$ is the non-electric energy; $l(t)$ is a continuous function describing the labour; α is an elasticity of substitution between capital and labour; β is an elasticity of substitution between electricity and non-electric energy; μ is the coefficient of depreciation; ρ is an elasticity of substitution between capital-labour and energy bundle; a and b are scale productivity factors.

By taking into account an economic sense, the control parameters can not exceed certain values. Together with the fact that the characteristics of regions nonnegative, this leads to restrictions of the form:

$$0 < a_u \leq u \leq b_u, \quad 0 < a_v \leq v \leq b_v, \quad 0 < a_w \leq w \leq b_w. \tag{2}$$

The functions $u(\cdot)$, $v(\cdot)$ and $w(\cdot)$ satisfying the relations (2) will be called admissible control. Let us introduce a vector-function $c(\cdot) = (u(\cdot), v(\cdot), w(\cdot))^\top$ and denote a set of the admissible controls by the symbol U.

We suppose that the parameters of regions are known at the initial moment t_0, which are positive by their economic sense, i.e. the initial state is given

$$\begin{gathered} Y(t_0) = Y^0, \quad E(t_0) = E^0, \quad K(t_0) = K^0, \quad N(t_0) = N^0, \\ Y^0, E^0, K^0, N^0 > 0. \end{gathered} \tag{3}$$

3 Main Results

In the present paper we consider an optimal control problem of the system (1)–(3). Further, we assume $t_0 = 0$.

Problem P1. It is required to determine the functions $Y^*(\cdot)$, $K^*(\cdot)$, $E^*(\cdot)$, $N^*(\cdot)$, solving the extremal problem

$$\max_{Y,K,E,N,c} J(Y, K, E, N, c), \tag{4}$$

$$J(Y, K, E, N, c) = \int_0^T d(t) \ln C(t)\, dt, \tag{5}$$

satisfying the system (1) and ensuring the fulfillment of restrictions (2). Consumption $C(t)$ at the moment t is determined by the classical formula [1]:

$$C = Y - I - f - G(E, N). \tag{6}$$

Here $d(t)$ is the coefficient which represents a social discount factor and an economic loss factor due to the impact of climate change; $I(t)$ is the investment used to built capital stock, we assume $I(t) = u(t)$; $f(t)$ is a net exports, $G(E(t), N(t))$ is the energy costs that represent the total costs of extracting resources and supplying electric and non-electric energy determined by the equality

$$G(E, N) = gE + hN, \tag{7}$$

where the positive coefficients g and h characterize the production cost of electricity and non-electric energy respectively. The functions $d(t)$ and $f(t)$ are assumed to be given, $d(t) > 0$ for $t \in [0, T]$.

The variable K does not appear explicitly in the definition of the maximized functional (5) and the equations for the characteristics Y, E and N, therefore the optimal trajectories $K(t)$ are determined only by the initial conditions K^0 and the optimal controls u^*. Thus, we can ignore the corresponding equation of the system (1) in solving the extremal problem (4).

Because of the economic sense of parameters of the system (1) let us impose the restrictions of the following form:

$$0 < \alpha, \beta < 1, \quad \rho < 0, \quad \mu > 0, \quad a_u, b_u, a_v, b_v, a_w, b_w > 0. \tag{8}$$

Lemma 3.1 *For the functions Y, E and N the following estimates are valid:*

$$Y(t) \geq Y_m(t_Y), \quad E(t) \leq E_M(t_E), \quad N(t) \leq N_M(t_N), \quad t \in [0, T],$$

where

$$\begin{aligned} Y_m(t) &= e^{-\mu t} Y^0 + \mu^{-1} \xi (1 - e^{-\mu t}), \\ E_M(t) &= e^{-\mu t} E^0 + \mu^{-1} b_v (1 - e^{-\mu t}), \\ N_M(t) &= e^{-\mu t} N^0 + \mu^{-1} b_w (1 - e^{-\mu t}), \\ \xi &= (a (a_u)^{\rho\alpha} \min_{\tau \in [0,T]} l^{\rho(1-\alpha)}(\tau) + b (a_v)^{\rho\beta} (a_w)^{\rho(1-\beta)})^{1/\rho}, \end{aligned} \tag{9}$$

$$t_Y = \begin{cases} 0, & \xi \geq \mu Y^0, \\ T, & \xi < \mu Y^0, \end{cases} \; t_E = \begin{cases} T, & b_v \geq \mu E^0, \\ 0, & b_v < \mu E^0, \end{cases} \; t_N = \begin{cases} T, & b_w \geq \mu N^0, \\ 0, & b_w < \mu N^0. \end{cases} \tag{10}$$

P r o o f. Let us write the Cauchy formula for equations of the system (1), then we obtain

$$\begin{aligned} Y(t) &= e^{-\mu t} Y^0 + \int_0^t e^{\mu(\tau - t)} (a u^{\rho\alpha}(\tau) l^{\rho(1-\alpha)}(\tau) + b v^{\rho\beta}(\tau) w^{\rho(1-\beta)}(\tau))^{1/\rho} \, d\tau, \\ E(t) &= e^{-\mu t} E^0 + \int_0^t e^{\mu(\tau - t)} v(\tau) \, d\tau, \quad N(t) = e^{-\mu t} N^0 + \int_0^t e^{\mu(\tau - t)} w(\tau) \, d\tau. \end{aligned} \tag{11}$$

The functions $E(t)$ and $N(t)$ take the maximum values for $v(t) \equiv b_v$ and $w(t) \equiv b_w$. In accordance to restrictions (8) the function $Y(t)$ takes the minimum values for the controls $u(t) \equiv a_u$, $v(t) \equiv a_v$ and $w(t) \equiv a_w$. Thus, by using the formulas (11) we find expressions for the lower bounds of the functions $Y(t) \geq$

$Y_m(t)$ and the upper bounds of the functions $E(t) \leq E_M(t)$ and $N(t) \leq N_M(t)$ of the form (9).

Let us differentiate the first equation of the system (9), the we have

$$\dot{Y}_m(t) = e^{-\mu t}(\xi - \mu Y^0).$$

The sign of the derivative is defined by the following relations: if $\xi \geq \mu Y^0$, then $\dot{Y}_m(t) \geq 0$, and if $\xi < \mu Y^0$, then $\dot{Y}_m(t) < 0$. So the function $Y_m(t)$ will take its minimum value at the time moment t_Y.

By differentiating the second and the third equations of the system (9), we obtain

$$\dot{E}_M(t) = e^{-\mu t}(b_v - \mu E^0), \quad \dot{N}_M(t) = e^{-\mu t}(b_w - \mu N^0).$$

By investigating the signs of derivatives in the similar way we have that the functions $E_M(t)$ and $N_M(t)$ will take maximum values at the time moments t_E and t_N respectively. □

Theorem 3.2 *Let the parameters of the system* (1) *satisfy the conditions* (8)*, the restrictions on the control* (2) *and the initial values* (3) *satisfy the inequalities*

$$Y_m(t_Y) - b_u - b_f - gE_M(t_E) - hN_M(t_N) > 0,$$

where Y_m, E_M and N_M are defined by the formulas (9)*, t_Y, t_E and t_N — by the formulas* (10)*; then the function of consumption takes only positive values*

$$C(t) > 0, \quad t \in [0, T].$$

P r o o f. By taking into account the function of energy expenditures (7) the following estimate for the function of consumption (6) is valid

$$C(t) = Y(t) - u(t) - f(t) - gE(t) - hN(t) \geq Y(t) - b_u - b_f - gE(t) - hN(t).$$

By using the inequalities of Lemma we obtain

$$C(t) \geq Y_m(t_Y) - b_u - b_f - gE_M(t_E) - hN_M(t_N).$$ □

Further we assume that the statement of the Theorem is valid. Let us reduce the system (1), so we subtract the third and the fourth equations multiplied by g and h respectively from the first equation. Then by introducing notations $Z = Y - gE - hN$ we reduce the system (1) of the form

$$\dot{Z}(t) = -\mu Z(t) + (au^{\rho\alpha}(t)l^{\rho(1-\alpha)}(t) + bv^{\rho\beta}(t)w^{\rho(1-\beta)}(t))^{1/\rho} - gv(t) - hw(t) \quad (12)$$

with a corresponding boundary condition

$$Z(0) = Z^0 = Y^0 - gE^0 - hN^0. \quad (13)$$

As a result we obtain the following optimal control problem.

Problem P2. It is required to define functions $Z^*(\cdot)$, $c^*(\cdot)$, solving the extremal problem

$$\max_{Z,c} J(Z,c), \quad J(Z,c) = \int\limits_0^T d(t)\ln(Z(t) - u(t) - l(t))\,dt,$$

satisfying the equations (12) with the boundary conditions (13) and ensuring the implementation of restrictions (2).

3.1 To the solution of the problem P2

We use the Pontryagin maximum principle [6] for investigation of properties of the vector function $x(t)$, $0 \le t \le T$, which is an optimal program control for the Problem P2.

Let us write the Hamilton–Pontryagin function for the problem, assuming that $\psi_0 = -1$:

$$H(t, Z, \psi, c) = \psi(-\mu Z + f^{1/\rho} - gv - hw) + d(t)\ln(Z - u - l(t)), \tag{14}$$

where $f = au^{\rho\alpha} l^{\rho(1-\alpha)}(t) + bv^{\rho\beta} w^{\rho(1-\beta)}$. Then we find the partial derivative of the hamiltonian with respect to coordinate$Z$:

$$\partial H/\partial Z = -\mu\psi + d(t)/(Z - u - l(t)).$$

So the conjugate equation $\dot{\psi} = -H'_Z$ takes the form

$$\dot{\psi} = \mu\psi - d(t)/(Z - u - l(t)), \tag{15}$$

The right end of the trajectory is free, therefore the conjugate variable satisfy the transversality condition

$$\psi(T) = 0. \tag{16}$$

By using the equation (12) with the condition (13), as well as (15) and (16), we obtain the boundary value problem of the maximum principle for the Problem P2 of the form

$$\begin{aligned} &\dot{Z} = -\mu Z + f^{1/\rho} - gv - hw, \quad && Z(t_0) = Z^0, \\ &\dot{\psi} = \mu\psi - d(t)/(Z - u - l(t)), && \psi(T) = 0. \end{aligned} \tag{17}$$

Assertion 3.3 *Let the assumptions of Theorem hold. Then the patrial derivative of the hamiltonian* (14) *with respect to control u is not equal to zero.*

P r o o f. Let us find the derivative of the hamiltonian (14) with respect the control u:

$$\partial H/\partial u = a\alpha\psi f^{1/\rho-1}u^{\alpha\rho-1}l^{\rho(1-\alpha)}(t) - d(t)/(Z-u-l(t)).$$

We shall prove by contradiction. Let $\partial H/\partial u = 0$, then by equating the found derivative to zero, we have

$$d(t)/(Z-u-l(t)) = a\alpha\psi f^{1/\rho-1}u^{\alpha\rho-1}l^{\rho(1-\alpha)}(t).$$

Let us substitute the previous expression into the conjugate system of the boundary value problem (17), in this case we obtain

$$\dot{\psi} = (\mu - a\alpha f^{1/\rho-1}u^{\alpha\rho-1}l^{\rho(1-\alpha)}(t))\psi, \quad \psi(T) = 0.$$

The solution of the latest system is defined by the formula

$$\psi = D\exp\left(\int_0^t (\mu - a\alpha f^{1/\rho-1}(\tau)u^{\alpha\rho-1}(\tau)l^{\rho(1-\alpha)}(\tau))\,d\tau\right).$$

By taking into account the boundary condition $\psi(T) = 0$, we find $D = 0$; then $\psi(t) \equiv 0$. This equality gives

$$d(t)/(Z(t)-u(t)-l(t)) \equiv 0,$$

which contradicts the conditions of positivity $C(t)$ and $d(t)$. □

This assertion follows two cases: either $u^*(t) = a_u$ or $u^*(t) = b_u$, $t \in [0,T]$. Let us find a derivative of the hamiltonian with respect to controls v and w:

$$\partial H/\partial v = \psi(b\beta(au^{*\rho}l^{\rho(1-\alpha)}(t) + bv^{\rho\beta}w^{\rho(1-\beta)})^{1/\rho-1}v^{\beta\rho-1}w^{\rho(1-\beta)} - g),$$
$$\partial H/\partial w = \psi(b(1-\beta)(au^{*\rho}l^{\rho(1-\alpha)}(t) + bv^{\rho\beta}w^{\rho(1-\beta)})^{1/\rho-1}v^{\beta\rho}w^{\rho(1-\beta)-1} - h).$$

By equating the derivatives to zero, we obtain

$$v^*(t) = a^{1/\rho}u^*(t)l^{1-\alpha}(t)\tilde{b}, \quad w^*(t) = a^{1/\rho}u^*(t)l^{1-\alpha}(t)(1-\beta)g\tilde{b}/(\beta h), \tag{18}$$

where $\tilde{b} = (g/(b\beta))^{1/(1-\rho)}(\beta h/((1-\beta)g))^{\rho(1-\beta)/(1-\rho)} - b((1-\beta)g/(\beta h))^{1-\beta}$.

3.2 An algorithm of solving the Problem P1

An analytical solution of the problem P1 as an optimal control problem is complicated by the nonlinearity of the system (1), as well as nonconvexity of the functional (5). For the solution of this problem methods of nonconvex optimization may be used [7, 8]. In this case we shall construct an optimal control investigating the restrictions (2).

Let t_i, $i = 0, \dots, n$, $t_n = T$, is a partition of the time interval $[t_0, T]$ with a step $\delta = (T - t_0)/n$. We assume $u^{(0)} = 0$, $v^{(0)} = 0$ and $w^{(0)} = 0$. On the i-th iteration the optimal control $u^{(i)}$ according to the Lemma 1 can be set to either a_u or b_u. For the optimal controls $v^{(i)}$ and $w^{(i)}$ we assume that they can be set to either a_v or b_v and either a_w or b_w respectively or values defined by the formulas (18).

For each set of the control parameters $c^{(1)}, \dots, c^{(n)}$ we solve the system of differential equations (1) with a help of Euler method [9]. Then on $(i+1)$-th step of the iteration procedure we obtain

$$\begin{aligned}
&Y^{(i+1)} = Y^{(i)} + \delta(-\mu Y^{(i)} + (a u^{(i+1)\alpha\rho} l^{(i)\rho(1-\alpha)} + b v^{(i+1)\beta\rho} w^{(i+1)\rho(1-\beta)})^{1/\rho}),\\
&K^{(i+1)} = K^{(i)} + \delta(-\mu K^{(i)} + u^{(i+1)}),\\
&E^{(i+1)} = E^{(i)} + \delta(-\mu E^{(i)} + v^{(i+1)}),\\
&N^{(i+1)} = N^{(i)} + \delta(-\mu N^{(i)} + w^{(i+1)}), \quad i = 0, \dots, n-1.
\end{aligned}$$

By using the found values of functions $Y^{(1)}, \dots, Y^{(n)}$, $K^{(1)}, \dots, K^{(n)}$, $E^{(1)}, \dots, E^{(n)}$, $N^{(1)}, \dots, N^{(n)}$, we obtain the values of functional J (5). For this we calculate the integral using the rectangle method [9]:

$$J = \delta \sum_{i=1}^{n} d^{(i)} \ln C^{(i)},$$

where $C^{(i)} = Y^{(i)} - u^{(i)} - l^{(i)} - G^{(i)}(E^{(i)}, N^{(i)})$.

Among the obtained values of the functional we choose the largest and emphasize the corresponding set of controls, which will be optimal.

4 Results of the numerical modeling

By using the described in previous section algorithm let us construct solution of the problem P1 for specific values of parameters of the system (1) and restrictions (2). We take Russian Federation and Ukraine as considered regions. This regions have the following set of parameters: $\alpha = 0.3$, $\beta = 0.45$, $\mu = 0.05$, $\rho = -1.5$ [1]. We investigate the dynamics of the main indexes on the time interval from 2010 to 2020 y.; the initial state — $Y^0 = 0.306$ trillion \$, $K^0 = 0.857$ trillion \$, $E^0 = 0.2$ TkWh, $N^0 = 5.258$ EJ [10]. Hereinafter \$ is USD2005.

Parameters of the energy expenditures function (7) are defined by equalities $h = 0.0025$ trillion \$/EJ, $g = 0.0563$ trillion \$/TkWh [1].

The function of the difference between exports and imports $F(t)$ is defined by the formula $F(t) = 0.027$ trillion \$. The function $l(t)$ describing a labour productivity and measured in efficiency units is chosen as a constant: $l(t) = 1$. The discount factor of utility is defined by the formula $d(t) = 1 - 0.01(t - 2010)$. By using the formulas for the basic characteristics of region introduced in [1], we find out $a = 5.44$ and $b = 0.64$.

As boundary values for control u we choose the investment volumes of 1996 and 2010 years, i.e. $a_u = 0.005$ trillion \$, $b_u = 0.03$ trillion \$ [10]. We choose a lower bound of the control u on the basis of production growth of electricity in 2005–2010 years, the top bound — in 1987–1988 years [10]. As a result we have $a_v = 0.005$ TkWh and $b_v = 0.012$ TkWh. We choose the following bounds of the control w : $a_w = 0.01$ EJ and $b_w = 0.15$ EJ. We choose a step of the time interval mesh equal to one year, i.e. $n = 10$.

Graphics of the main macroeconomic characteristics of the region are presented at fig. 1–3. The results obtained by the algorithm of solving the problem P1 are presented by the black solid line. Data of the basic scenario of the model MERGE is shown by the dotted line. At fig. 1 a GDP forecast in Ukraine given by the International Monetary Fund [11] is presented.

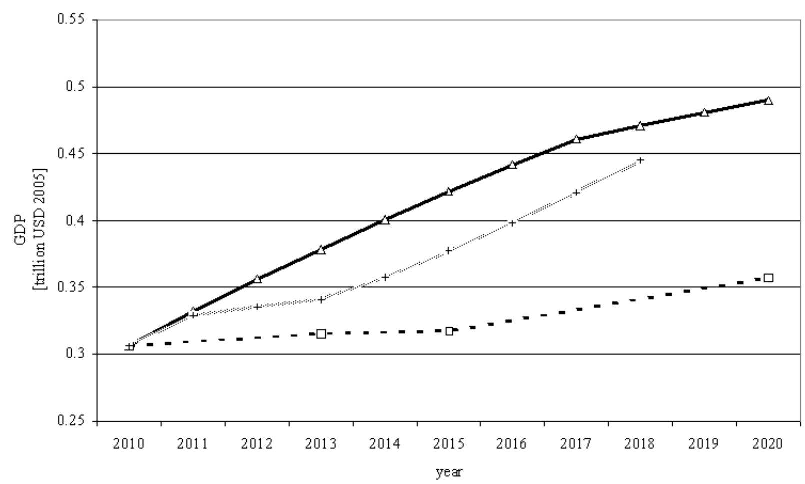

Fig. 1. Realized GDP

Investments taken as the control parameter u reach the maximum b_u from 2010 to 2018 years, then take the value a_u up to 2020 year.

In the considering problem for the given set of parameters the optimal control u^* has a single switch point. The optimal control of production of energy is defined by the formulas $v^*(t) = b_v$, $w^*(t) = b_w$.

Graphics of electricity and nonelectric energy of the basic scenario of MERGE and graphics obtained by the numeric solution of the problem P1 demonstrate a similar behavior. Essential distinctions are observed at the GDP graphic. Differences from IMF forecast can be explained that data of 2011 and 2012 years is used in this forecast whereas we use only data at the initial moment (2010 y.).

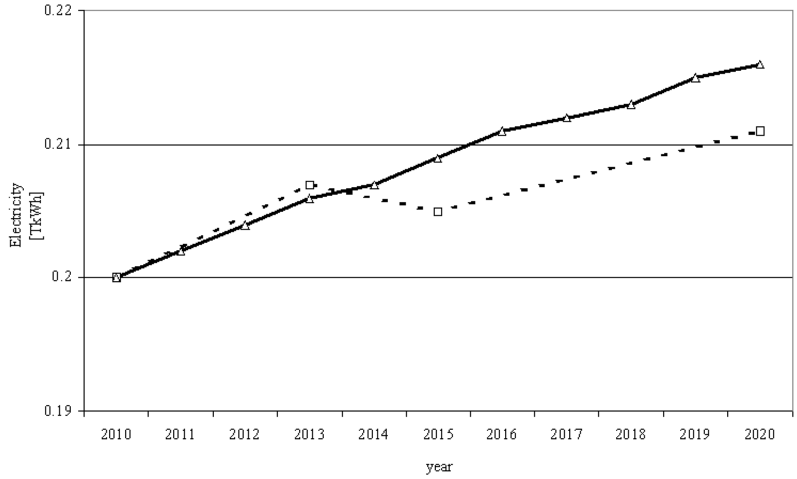

Fig. 2. Production of electricity

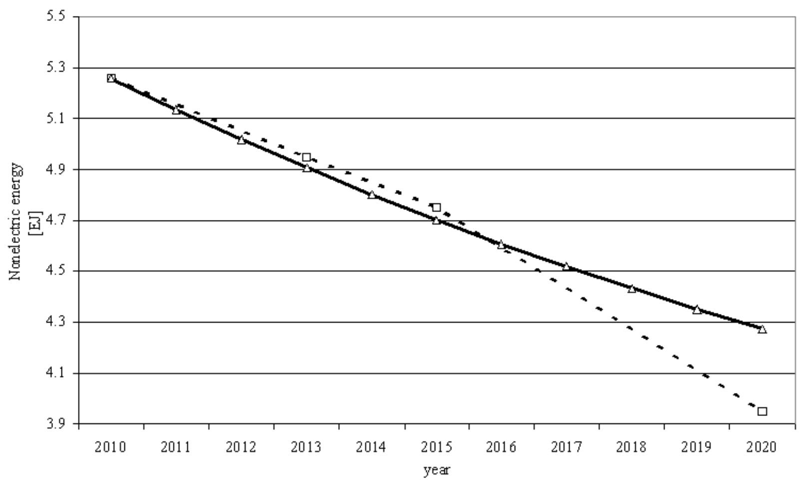

Fig. 3. Production of nonelectric energy

When building scenarios in MERGE the method of intertemporal optimization is used, as well as such characteristics of the region, as a potential GDP in the whole time period, which probably causes the observed differences with the results of numerical solution of the problem P1.

Concluding remarks

In conclusion we note that the region is considered as a separate system with a given external influence whereas the cumulative impact of regions on each other in MERGE is taken into account. Therefore one of the main directions of research is considering several interacting regions.

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