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Asymptotics of Regularized Solutions of an Ill-Posed Cauchy Problem for an Autonomous Linear System of Differential Equations with Commensurable Delays

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Abstract—For an autonomous linear system of differential equations with commensurable delays, asymptotic formulas are found that describe the analytic dependences of regularized solutions of the system on the regularization parameter. The problem is solved under the requirement that the initial function is sufficiently smooth but with the violation of the conditions that guarantee the continuous extension of solutions in the direction of decreasing time.
Keywords: differential equations with delay, ill-posed problem, asymptotic methods.

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INTRODUCTION

We consider an autonomous linear system of differential equations with commensurable delays

$$\frac{dx(t)}{dt} = \sum_{j=0}^{m} A_j x(t - j\tau), \tag{0.1}$$

where $x\colon \mathbb{R} \to \mathbb{R}^n$, A_j for $j = [0 : m]$ are constant $n \times n$ matrices, $\tau > 0$, and $m \geq 2$.

The problem of finding a solution of the Cauchy problem for system (0.1) on any interval of the positive half-line is well-posed for an initial function $\varphi \in C = C([-r, 0], \mathbb{R}^n)$, where $r = m\tau$ [1, Sect. 6.4; 2, p. 17; 3, Sect. 2.2]. For finding its solution $x(\cdot, \varphi)$ on the positive half-line, we can use a step-by-step procedure, which, in the functional state space C [4, p. 182], is described by the formulas

$$x_k = Ux_{k-1}, \quad k \geq 1, \quad x_0 = \varphi, \quad x_k(\cdot) = x(\cdot + kr). \tag{0.2}$$

Here, $U\colon C \to C$ is a linear completely continuous operator defined by the formula

$$(U\varphi)(\theta) = V(r + \theta)\varphi(0) + \sum_{j=1}^{m} \int\limits_{0}^{j\tau} V(r + \theta - s)A_j \varphi(s - j\tau)\, ds, \quad \theta \in [-r, 0], \tag{0.3}$$

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where V is the Cauchy matrix of (0.1). This formula follows from [5, formula (2.1)] if the system is autonomous and $\omega = r$, see also [3, 6]. Consecutive iterations localize intervals of solution of (0.1) on the positive half-line by the formulas $x(kr + \theta, \varphi) = x_k(\theta)$, where $\theta \in [-r, 0]$ and $k \geq 1$.

To find a solution of the Cauchy problem $x(\cdot, \varphi)$ on the negative half-line, we use step-by-step procedure (0.2) for negative values of the index k. In this procedure, consecutive iterations are defined by the formulas

$$x_{k,\alpha} = R(x_{k+1,\alpha}, \alpha), \quad k \leq -1, \quad x_{0,\alpha} = \varphi,$$

where R is a regularizing operator of the equation

$$Ux = \varphi \tag{0.4}$$

and α is a regularization parameter. We define a regularized solution $x(\cdot, \varphi)$ of the Cauchy problem on the negative half-line by the formulas $x(kr + \theta, \varphi) = x_{k,\alpha}(\theta)$, where $k \leq -1$ and $\theta \in (-r, 0]$.

Operator (0.3) can be continuously extended to the separable Hilbert space $H = L_2([-r, 0), \mathbb{R}^n) \times \mathbb{R}^n$ with inner product $(\varphi, \psi) = \psi^\top(0)\varphi(0) + \int_{-r}^{0} \psi^\top(s)\varphi(s)\, ds$. This extension preserves the complete continuity of the operator U.

To find a solution of ill-posed problem (0.4), we use Tikhonov's regularization method [7, Ch. 3] with a stabilizing functional of the following form:

$$\Omega[x] = x^\top(0)x(0) + \int_{-r}^{0} (x^\top(s)x(s) + x'^\top(s)x'(s))\, ds, \quad x \in W_2^1([-r, 0], \mathbb{R}^n). \tag{0.5}$$

In the case when U is an integral operator in the space $L_2([-r, 0], \mathbb{R}^n)$, ill-posed problem (0.4) with stabilizing functional (0.5) has a unique solution that satisfies a boundary value problem for an integro-differential equation [8]. A similar result can be proved for operator (0.3) in the space H.

In [9], for an autonomous linear system of differential equations with one delay, we established the equivalence of a boundary value problem for an integro-differential equation and a boundary value problem for a system of ordinary differential equations and constructed an asymptotics for regularized solutions of an ill-posed Cauchy problem for smooth initial functions. We generalized the results obtained in [9] to nonautonomous linear differential equations with one delay [10]; to nonlinear differential equations with a delay that describes a population change [11]; and, in the present paper, to autonomous linear systems of differential equations with commensurable delays.

1. BOUNDARY VALUE PROBLEM FOR A REGULARIZED SOLUTION OF EQUATION (0.4)

In finding a regularized solution of equation (0.4) for a fixed value of the regularization parameter α, it is required to find an element $x_\alpha \in H$ that minimizes the functional

$$M^\alpha[\varphi, x] = (Ux - \varphi, Ux - \varphi) + \alpha\Omega[x].$$

This element satisfies the boundary value problem for integro-differential equation [9]

$$\begin{aligned} &(U^*(Ux - \varphi))(\vartheta) + \alpha\big(x(\vartheta) - x''(\vartheta)\big) = 0, \quad \vartheta \in [-r, 0), \\ &(U^*(Ux - \varphi))(0) + \alpha\big(x(0) + x'(0)\big) = 0, \quad x'(-r) = 0. \end{aligned} \tag{1.1}$$

Let us show that, for an autonomous linear system with aftereffect of the general form

$$\frac{dx(t)}{dt} = \int_{-r}^{0} [d\eta(s)]\, x(t+s), \tag{1.2}$$

where $\eta\colon [-r,0] \to \mathbb{R}^{n\times n}$ is a function of bounded variation, boundary value problem (1.1) can be replaced by an equivalent problem for a system of functional differential equations.

Assertion 1. *Let* $\det A_m \neq 0$ *and* $L\varphi \in H$. *Then, a regularized solution of equation* (1.2) *coincides with the* x *component of a solution of the system of functional differential equations*

$$\alpha\frac{d^2x(\theta)}{d\theta^2} = \alpha x(\theta) + \frac{d}{d\theta}\int_{-r}^{\theta} \Big(\eta^\top(\theta-t-r) - \eta^\top(-r)\Big)\left(\hat{\chi}(t) - \hat{z}(t)\right)\, dt, \tag{1.3}$$

$$\frac{d\hat{\chi}(\theta)}{d\theta} = \int_{\theta}^{0} [d_t\eta^\top(\theta-t)]\, \hat{\chi}(t) - \chi(\theta), \tag{1.4}$$

$$\frac{d\hat{z}(\theta)}{d\theta} = \int_{\theta}^{0} [d_t\eta^\top(\theta-t)]\, \hat{z}(t) - \varphi(\theta), \tag{1.5}$$

$$\frac{d\chi(\theta)}{d\theta} = \int_{-r-\theta}^{0} [d\eta(t)]\, \chi(\theta+t) + \int_{\theta}^{0} [d_t\eta(t-\theta-r)]\, x(t) \tag{1.6}$$

with boundary conditions

$$\alpha(x(0) + x'(0)) + \hat{\chi}(-r) + \hat{z}(-r) = 0, \tag{1.7}$$

$$\hat{\chi}(0) = \chi(0), \quad \hat{z}(0) = \varphi(0), \tag{1.8}$$

$$\chi(-r) = x(0), \quad x'(-r) = 0. \tag{1.9}$$

Here, a positive number α *is the regularization parameter of the ill-posed problem.*

Proof. For system (1.2), the operator U is defined by the formula

$$(Ux)(\theta) = V(\theta+r)x(0) + \int_{-r}^{-0} \frac{\partial}{\partial s}\bigg(\int_{-r}^{\theta} V(\theta-z)\eta(s-z-r)\, dz\bigg) x(s)\, ds, \quad \theta \in [-r,0],$$

where V is the Cauchy matrix of (1.2) [5]. The adjoint operator $U^*\colon H \to H$ is defined by the formulas

$$(U^*y)(0) = V^\top(r)y(0) + \int_{-r}^{0} V^\top(s+r)y(s)\, ds,$$

$$(U^*y)(\theta) = \frac{d}{d\theta}\bigg(\int_{-r}^{0} \eta^\top(\theta-z-r)V^\top(-z)\, dz\bigg) y(0)$$

$$+ \int_{-r}^{0} \frac{d}{d\theta}\bigg(\int_{-r}^{s} \eta^\top(\theta-z-r)V^\top(s-z)\, dz\bigg) y(s)\, ds, \quad \theta \in [-r,0).$$

Let us introduce auxiliary functions χ, $\hat{\chi}$, and $\hat{z}$:

$$\chi(\vartheta) = (Ux)(\vartheta), \quad \hat{\chi}(\vartheta) = (U_1^*\chi)(\vartheta), \quad \hat{z}(\vartheta) = (U_1^*\varphi)(\vartheta), \quad \vartheta \in [-r, 0),$$
$$\chi(0) = \chi(-0), \quad \hat{\chi}(0) = (U_1^*\chi)(-0), \quad \hat{z}(0) = (U_1^*\varphi)(-0),$$

where the operator $U_1^*\colon H \to H^1 = W_2^1([-r,0),\mathbb{R}^n) \times \mathbb{R}^n$ is defined by the formulas

$$(U_1^*y)(\theta) = V^\top(-\theta)y(0) + \int\limits_\theta^0 V^\top(s-\theta)y(s)\,ds, \quad \theta \in [-r,0), \quad (U_1^*y)(0) = 0.$$

Then, the boundary value problem can be written in the form

$$(U_2^*(\hat{\chi} - \hat{z}))(\vartheta) + \alpha\big(x(\vartheta) - x''(\vartheta)\big) = 0, \quad \vartheta \in [-r, 0), \tag{1.10}$$

$$(U^*(\chi - \varphi))(0) + \alpha\big(x(0) + x'(0)\big) = 0, \quad x'(-r) = 0, \tag{1.11}$$

where the operator $U_2^*\colon H^1 \to H$ is defined by the formulas

$$(U_2^*y)(\theta) = \frac{d}{d\theta}\int\limits_{-r}^\theta \big(\eta(\theta - s - r) - \eta(-r)\big)^\top y(s)\,ds, \quad \theta \in [-r, 0), \quad (U_2^*y)(0) = 0.$$

In view of the definition of the operator U_2^*, equation (1.10) is reduced to form (1.3).

Using the definitions of the operator U_1^* and the Cauchy matrix V, we replace the integral dependence between the variables $\hat{\chi}$ and χ by the boundary value problem

$$\frac{d\hat{\chi}(\theta)}{d\theta} = \frac{\partial V^\top(-\theta)}{\partial\theta}\chi(0) - \chi(\theta) + \int\limits_\theta^0 \frac{\partial V^\top(s-\theta)}{\partial\theta}\chi(s)\,ds, \quad \theta \in [-r, 0), \quad \hat{\chi}(-0) = \chi(0).$$

Then, in view of the properties of the function V [3, Sect. 6.2], we have

$$\frac{\partial V^\top(-\theta)}{\partial\theta}\chi(0) + \int\limits_\theta^0 \frac{\partial V^\top(s-\theta)}{\partial\theta}\chi(s)\,ds = \int\limits_\theta^0 [d_\alpha\eta^\top(\theta-\alpha)]\left(\int\limits_\alpha^0 V^\top(s-\alpha)\chi(s)\,ds + V^\top(-\alpha)\chi(0)\right)$$
$$= \int\limits_\theta^0 [d_\alpha\eta^\top(\theta-\alpha)]\,\hat{\chi}(\alpha), \quad \theta \in [-r, 0).$$

As a result, we obtain equation (1.4) and the former condition in (1.8).

Applying similar calculations, we transform the dependence between the variables $\hat{z}$ and φ to a boundary problem, which is differential equation (1.5) with the latter boundary condition in (1.8).

Using the definitions of the operator U and the Cauchy matrix V, we replace the integral dependence between the variables χ and x by boundary value problem (1.6) with the former boundary condition in (1.9).

Transforming the first term in boundary condition (1.11), we obtain

$$U^*(\chi - \varphi)(0) = V^\top(-r)(\chi(0) - \varphi(0)) + \int\limits_{-r}^0 V^\top(s+r)(\chi(s) - \varphi(s))\,ds$$

$$= U_1^*(\chi - \varphi)(-r) = \hat{\chi}(-r) - \hat{z}(-r).$$

Consequently, boundary condition (1.11) coincides with boundary condition (1.7). □

Transform boundary value problem (1.3)–(1.9) using the special form of η for system (0.1)

$$\eta(0) = 0, \quad \eta(s) = -\sum_{j=0}^{k} A_j, \quad -(k+1)\tau < s < -k\tau, \quad 0 \le k \le m-1, \quad \eta(-m\tau) = -\sum_{j=0}^{m} A_j$$

and the special notation

$$\hat{x}_j(s) = x(s-(j-1)\tau), \quad \hat{\chi}_j(s) = \hat{\chi}(s-(j-1)\tau), \quad \hat{z}_j(s) = \hat{z}(s-(j-1)\tau),$$
$$\chi_j(s) = \chi(s-(j-1)\tau), \quad \varphi_j(s) = \varphi(s-(1-j)\tau), \quad s \in [-\tau, 0], \quad j = [1:m].$$

Theorem 1. *Let* $\det A_m \ne 0$ *and* $\varphi \in H$. *Then, the formulas* $x(\theta) = \hat{x}_j(\theta + (j-1)\tau)$, $\theta \in [-j\tau, -(j-1)\tau]$, $j = [1:m]$, *define a regularized solution of equation* (0.4). *Here, the functions* $\hat{x}_j$, $j = [1:m]$, *are components of a solution of the system of ordinary differential equations*

$$\alpha \hat{x}_j'' = \alpha \hat{x}_j + \sum_{i=j}^{m} A_{m+j-i}^{\top}(\hat{\chi}_i - \hat{z}_i), \tag{1.12}$$

$$\hat{\chi}_j' = -\sum_{k=1}^{j} A_{j-k}^{\top}\hat{\chi}_k - \chi_j, \tag{1.13}$$

$$\hat{z}_j' = -\sum_{k=1}^{j} A_{j-k}^{\top}\hat{z}_k - \varphi_j, \tag{1.14}$$

$$\chi_j' = \sum_{k=j}^{m} A_{k-j}\chi_k + \sum_{k=1}^{j} A_{m+k-j}\hat{x}_k, \quad j = [1:m], \tag{1.15}$$

with boundary conditions

$$\hat{x}_j(-\tau) = \hat{x}_{j+1}(0), \quad \hat{x}_j'(-\tau) = \hat{x}_{j+1}'(0), \quad j = [1:m-1], \quad \hat{x}_m'(-\tau) = 0, \tag{1.16}$$

$$\alpha(\hat{x}_1(0) + \hat{x}_1'(0)) + \hat{\chi}_m(-\tau) - \hat{z}_m(-\tau) = 0, \tag{1.17}$$

$$\hat{\chi}_j(-\tau) = \hat{\chi}_{j+1}(0), \quad \hat{z}_j(-\tau) = \hat{z}_{j+1}(0), \quad j = [1:m-1], \tag{1.18}$$

$$\hat{\chi}_1(0) = \chi_1(0), \quad \hat{z}_1(0) = \varphi_1(0), \tag{1.19}$$

$$\chi_j(-\tau) = \chi_{j+1}(0), \quad j = [1:m-1], \quad \chi_m(-\tau) = \hat{x}_1(0). \tag{1.20}$$

Proof. Boundary conditions (1.7)–(1.9) and the continuity of solutions of system of functional differential equations (1.3)–(1.6) imply the validity of boundary conditions (1.16)–(1.20). Setting $\theta = s - (j-1)\tau$, where $s \in [-\tau, 0]$ and $j = [1:m]$, we transform system of equations (1.3)–(1.6) to the form

$$\alpha \hat{x}_j''(s) = \alpha \hat{x}_j(s)$$

$$+ \frac{d}{ds} \int\limits_{-m\tau}^{s-(j-1)\tau} \Big(\eta^{\top}(t) - \eta^{\top}(-m\tau)\Big)\, (\hat{\chi}(s-(m+j-1)\tau - t) - \hat{z}(s-(m+j-1)\tau - t))\, dt,$$

$$\hat{\chi}_j'(s) = -\int\limits_{s-(j-1)\tau}^{0} [d\eta^\top(t)]\,\hat{\chi}(s-(j-1)\tau-t) - \chi_j(s),$$

$$\hat{z}_j'(s) = -\int\limits_{s-(j-1)\tau}^{0} [d\eta^\top(t)]\,\hat{z}(s-(j-1)\tau-t) - \varphi_j(s),$$

$$\chi_j'(s) = \int\limits_{-(m-j+1)\tau-s}^{0} [d\eta(t)]\,\chi(s+t-(j-1)\tau) + \int\limits_{-m\tau}^{-(m-j+1)\tau-s} [d\eta(t)]\,x(s+t-(m+j-1)\tau),$$

$$s\in[-\tau,0],\quad j=[1:m].$$

Transforming the integral terms in these equations, we obtain

$$\frac{d}{ds}\int\limits_{-m\tau}^{s-(j-1)\tau} \Big(\eta^\top(t)-\eta^\top(-m\tau)\Big)\,(\hat{\chi}(s-(m+j-1)\tau-t)-\hat{z}(s-(m+j-1)\tau-t))\,dt$$

$$= \frac{d}{ds}\sum_{i=j}^{m} A_i^\top \int\limits_{-j\tau}^{s-(j-1)\tau} (\hat{\chi}(s-(m+j-1)\tau-t)-\hat{z}(s-(m+j-1)\tau-t))\,dt$$

$$+\frac{d}{ds}\sum_{k=j}^{m-1}\sum_{i=k+1}^{m} A_i^\top \int\limits_{-(k+1)\tau}^{-k\tau} (\hat{\chi}(s-(m+j-1)\tau-t)-\hat{z}(s-(m+j-1)\tau-t))\,dt$$

$$= \sum_{k=j}^{m-1}\sum_{i=k+1}^{m} A_i^\top(\hat{\chi}_{m+j-k-1}(s)-\hat{z}_{m+j-k-1}(s)-\hat{\chi}_{m+j-k}(s)+\hat{z}_{m+j-k}(s))$$

$$+\sum_{i=j}^{m} A_i^\top(\hat{\chi}_m(s)-\hat{z}_m(s)) = \sum_{i=j}^{m} A_i^\top(\hat{\chi}_{m+j-i}(s)-\hat{z}_{m+j-i}(s)) = \sum_{i=j}^{m} A_{m+j-i}^\top(\hat{\chi}_i(s)-\hat{z}_i(s)),$$

$$\int\limits_{s-(j-1)\tau}^{0} [d\eta^\top(t)]\,\hat{\chi}(s-(j-1)\tau-t) = \sum_{k=1}^{j} A_{j-k}^\top \hat{\chi}_k(s),$$

$$\int\limits_{s-(j-1)\tau}^{0} [d\eta^\top(t)]\,\hat{z}(s-(j-1)\tau-t) = \sum_{k=1}^{j} A_{j-k}^\top \hat{z}_k(s),$$

$$\int\limits_{-(m-j+1)\tau-s}^{0} [d\eta(t)]\,\chi(s+t-(j-1)\tau) = \sum_{k=j}^{m} A_{k-j}\chi_k(s),$$

$$\int\limits_{-m\tau}^{-(m-j+1)\tau-s} [d\eta(t)]\,x(s+t-(m+j-1)\tau) = \sum_{k=1}^{j} A_{m+k-j}\hat{x}_k(s),\quad s\in[-\tau,0],\quad j=[1:m].$$

These equalities imply the validity of the theorem. □

In the solution of ill-posed problem (0.4), the regularization parameter α can take arbitrarily small positive values. Hence, boundary value problem (1.12)–(1.20) is singular. We pose the problem of finding the dependence of the components $\hat{x}_j$, $j = [1 : m]$, of a solution of boundary value problem (1.12)–(1.20) for a system of ordinary differential equations on the regularization parameter α. For an autonomous linear system of differential equations with one delay, this problem was solved in [9].

2. TRANSFORMATION OF BOUNDARY VALUE PROBLEM (1.12)–(1.20)

Solving the problem stated in the preceding section, we transform boundary value problem (1.12)–(1.20) by eliminating the variables $\hat{\chi}_j$, $\hat{z}_j$, and χ_j, $j = [1 : m]$.

Theorem 2. *Suppose that* $\det A_m \neq 0$ *and* $\varphi \in W_2^1([-r, 0], \mathbb{R}^n)$. *Then, the formulas* $x(\theta) = \hat{x}_j(\theta + (j-1)\tau)$, *where* $\theta \in [-j\tau, -(j-1)\tau]$ *and* $j = [1 : m]$, *define a regularized solution of equation* (0.4). *Here, the functions* $\hat{x}_j$, $j = [1 : m]$, *are the components of a solution of the system of ordinary differential equations*

$$\hat{x}_j^{IV} - \hat{x}_j'' + \sum_{q=1}^{m} P_{jq}(\hat{x}_q''' - \hat{x}_q') - \sum_{q=1}^{m} Q_{jq}(\hat{x}_q'' - \hat{x}_q) + \alpha^{-1} \sum_{q=1}^{m} B_{jq}\hat{x}_q = \alpha^{-1} f_j(s), \tag{2.1}$$
$$s \in [-\tau, 0], \quad j = [1 : m],$$

with boundary conditions

$$\hat{x}_j(-\tau) = \hat{x}_{j+1}(0), \quad \hat{x}_j'(-\tau) = \hat{x}_{j+1}'(0), \quad j = [1 : m-1], \quad \hat{x}_m'(-\tau) = 0, \tag{2.2}$$

$$\sum_{k=1}^{m} C_{k-1}(\hat{x}_k'''(0) - \hat{x}_k'(0)) + (I + A_0^\top) \sum_{k=1}^{m} C_{k-1}(\hat{x}_k''(0) - \hat{x}_k(0)) = 0, \tag{2.3}$$

$$\begin{gathered} \hat{x}_j''(-\tau) - \hat{x}_{j+1}''(0) + A_j^\top(\hat{x}_1(0) + \hat{x}_1'(0)) = 0, \quad j = [1 : m-1], \\ \hat{x}_m''(-\tau) - x_m(-\tau) + A_m^\top(\hat{x}_1(0) + \hat{x}_1'(0)) = 0, \end{gathered} \tag{2.4}$$

$$\begin{gathered} \hat{x}_j'''(-\tau) - \hat{x}_{j+1}'''(0) + A_j^\top \sum_{k=1}^{m} A_{m-k}^\top \sum_{q=k}^{m} C_{q-k}(\hat{x}_q''(-\tau) - \hat{x}_q(-\tau)) \\ + \alpha^{-1} A_j^\top(\hat{x}_1(0) - \varphi_m(-\tau)) = 0, \quad j = [1 : m-1], \\ \hat{x}_m'''(-\tau) - \hat{x}_m'(-\tau) + A_m^\top \sum_{k=1}^{m} A_{m-k}^\top \sum_{q=k}^{m} C_{q-k}(\hat{x}_q''(-\tau) - \hat{x}_q(-\tau)) \\ + \alpha^{-1} A_m^\top(\hat{x}_1(0) - \varphi_m(-\tau)) = 0. \end{gathered} \tag{2.5}$$

Here, $f_j(s) = \sum_{i=j}^{m} A_{m+j-i}^\top\big(\varphi_i'(s) - \sum_{k=i}^{m} A_{k-i}\varphi_k(s)\big)$, *where* $s \in [-\tau, 0]$ *and* $j = [1 : m]$, *and the matrix coefficients of system of equations* (2.1) *are defined by the formulas*

$$\sum_{q=1}^{m} B_{jq}\hat{x}_q = \sum_{i=j}^{m} A_{m+j-i}^\top \sum_{q=1}^{i} A_{m+q-i}\hat{x}_q, \tag{2.6}$$

$$\sum_{q=1}^{m} Q_{jq}\hat{x}_q = \sum_{i=j}^{m} A_{m+j-i}^\top \sum_{k=i}^{m} A_{k-i} \sum_{p=1}^{k} A_{k-p}^\top \sum_{q=p}^{m} C_{q-p}\hat{x}_q,$$

$$\sum_{q=1}^{m} P_{jq}\hat{x}_q = \sum_{i=j}^{m} A^{\top}_{m+j-i} \sum_{k=1}^{i} A_{i-k} \sum_{q=k}^{m} C_{q-k}\hat{x}_q - \sum_{i=j}^{m} A^{\top}_{m+j-i} \sum_{k=i}^{m} A_{k-i} \sum_{q=k}^{m} C_{q-k}\hat{x}_q, \quad j = [1:m],$$

where the matrices C_i, $i = [0 : m-1]$, *are uniquely defined by the formulas*

$$C_0 = A_m^{-1\top}, \quad \sum_{i=0}^{p} A^{\top}_{m+i-p} C_i = 0, \quad p = [1 : m-1].$$

Proof. Introducing the new variables $y_j = \hat{\chi}_j - \hat{z}_j$, $j = [1 : m]$, we replace boundary value problem (1.12)–(1.20) by the following problem:

$$\alpha \hat{x}''_j = \alpha \hat{x}_j + \sum_{i=j}^{m} A^{\top}_{m+j-i} y_i, \tag{2.7}$$

$$y'_j = -\sum_{k=1}^{j} A^{\top}_{j-k} y_k - \chi_j + \varphi_j, \tag{2.8}$$

$$\chi'_j = \sum_{k=j}^{m} A_{k-j}\chi_k + \sum_{k=1}^{j} A_{m+k-j}\hat{x}_k, \quad j = [1 : m], \tag{2.9}$$

with boundary conditions

$$\hat{x}_j(-\tau) = \hat{x}_{j+1}(0), \quad \hat{x}'_j(-\tau) = \hat{x}'_{j+1}(0), \quad j = [1 : m-1], \quad \hat{x}'_m(-\tau) = 0, \tag{2.10}$$

$$y_1(0) = \chi_1(0) - \varphi_1(0), \tag{2.11}$$

$$y_j(-\tau) = y_{j+1}(0), \quad j = [1 : m-1], \quad \alpha(\hat{x}_1(0) + \hat{x}'_1(0)) + y_m(-\tau) = 0, \tag{2.12}$$

$$\chi_j(-\tau) = \chi_{j+1}(0), \quad j = [1 : m-1], \quad \chi_m(-\tau) = \hat{x}_1(0). \tag{2.13}$$

From (2.7) and (2.8), we find

$$\chi_j = -y'_j - \sum_{k=1}^{j} A^{\top}_{j-k} y_k + \varphi_j, \tag{2.14}$$

$$y_j = \alpha \sum_{k=j}^{m} C_{k-j}(\hat{x}''_k - \hat{x}_k), \quad j = [1 : m]. \tag{2.15}$$

Since $\varphi_j \in W_2^1([-\tau, 0], \mathbb{R}^n)$, by (2.8), we have $y_j \in W_2^2([-\tau, 0], \mathbb{R}^n)$. Then, taking into account (2.7), we obtain $\hat{x}_j \in W_2^4([-\tau, 0], \mathbb{R}^n)$. Further, substituting (2.15) into (2.14), we obtain

$$\chi_j = -\alpha \sum_{k=j}^{m} C_{k-j}(\hat{x}'''_k - \hat{x}'_k) - \alpha \sum_{k=1}^{j} A^{\top}_{j-k} \sum_{q=k}^{m} C_{q-k}(\hat{x}''_q - \hat{x}_q) + \varphi_j, \quad j = [1 : m]. \tag{2.16}$$

Hence, in view of equation (2.9), we find the system of ordinary differential equations

$$\alpha \sum_{k=j}^{m} C_{k-j}(\hat{x}^{IV}_k - \hat{x}''_k) + \alpha \sum_{k=1}^{j} A^{\top}_{j-k} \sum_{q=k}^{m} C_{q-k}(\hat{x}'''_q - \hat{x}'_q) - \alpha \sum_{k=j}^{m} A_{k-j} \sum_{q=k}^{m} C_{q-k}(\hat{x}'''_q - \hat{x}'_q)$$

$$-\alpha\sum_{k=j}^{m}A_{k-j}\sum_{p=1}^{k}A_{k-p}^{\top}\sum_{q=p}^{m}C_{q-p}(\hat{x}_q''-\hat{x}_q)+\sum_{k=1}^{j}A_{m+k-j}\hat{x}_k=\varphi_j'(s)-\sum_{k=j}^{m}A_{k-j}\varphi_k(s),$$

where $j=[1:m]$ and $s\in[-\tau,0]$. Using the definitions of the matrices C_i, $i=[0:m-1]$, we transform the latter system to the form

$$\begin{aligned}\hat{x}_j^{IV}-\hat{x}_j''+\sum_{i=j}^{m}A_{m+j-i}^{\top}\sum_{k=1}^{i}A_{i-k}^{\top}\sum_{q=k}^{m}C_{q-k}(\hat{x}_q'''-\hat{x}_q')-\sum_{i=j}^{m}A_{m+j-i}^{\top}\sum_{k=i}^{m}A_{k-i}\sum_{q=k}^{m}C_{q-k}(\hat{x}_q'''-\hat{x}_q')\\-\sum_{i=j}^{m}A_{m+j-i}^{\top}\sum_{k=i}^{m}A_{k-i}\sum_{p=1}^{k}A_{k-p}^{\top}\sum_{q=p}^{m}C_{q-p}(\hat{x}_q''-\hat{x}_q)+\alpha^{-1}\sum_{i=j}^{m}A_{m+j-i}^{\top}\sum_{q=1}^{i}A_{m+q-i}\hat{x}_q\\=\alpha^{-1}\sum_{i=j}^{m}A_{m+j-i}^{\top}\Big(\varphi_i'(s)-\sum_{k=i}^{m}A_{k-i}\varphi_k(s)\Big),\quad j=[1:m],\quad s\in[-\tau,0],\end{aligned}$$

which coincides in form with system of equations (2.1). Boundary conditions (2.10) coincide with (2.2). Applying formulas (2.15) and (2.16) to (2.11), we obtain boundary condition (2.3). Using formulas (2.15), we replace boundary conditions (2.12) by the following ones:

$$\begin{gathered}\sum_{k=j}^{m}C_{k-j}(\hat{x}_k''(-\tau)-\hat{x}_k(-\tau))=\sum_{k=j}^{m-1}C_{k-j}(\hat{x}_{k+1}''(0)-\hat{x}_{k+1}(0)),\quad j=[1:m-1],\\C_0(\hat{x}_m''(-\tau)-\hat{x}_m(-\tau))+\hat{x}_1(0)+\hat{x}_1'(0)=0.\end{gathered}\tag{2.17}$$

In view of these equalities, boundary conditions (2.13) take the form

$$\begin{gathered}\sum_{k=j}^{m}C_{k-j}(\hat{x}_k'''(-\tau)-\hat{x}_k'(-\tau))=\sum_{k=j}^{m-1}C_{k-j}(\hat{x}_{k+1}'''(0)-\hat{x}_{k+1}'(0)),\quad j=[1:m-1],\\C_0(\hat{x}_m'''(-\tau)-\hat{x}_m'(-\tau))+\sum_{k=1}^{m}A_{m-k}^{\top}\sum_{q=k}^{m}C_{q-k}(\hat{x}_q''(-\tau)-\hat{x}_q(-\tau))\\+\alpha^{-1}(\hat{x}_1(0)-\varphi_m(-\tau))=0.\end{gathered}\tag{2.18}$$

Using equalities (2.2) in boundary conditions (2.17) and (2.18), we obtain

$$\sum_{k=j}^{m-1}C_{k-j}(\hat{x}_k''(-\tau)-\hat{x}_{k+1}''(0))+C_{m-j}(\hat{x}_m''(-\tau)-\hat{x}_m(-\tau))=0,\quad j=[1:m-1],$$

$$C_0(\hat{x}_m''(-\tau)-\hat{x}_m(-\tau))+\hat{x}_1(0)+\hat{x}_1(0)=0,$$

$$\sum_{k=j}^{m-1}C_{k-j}(\hat{x}_k'''(-\tau)-\hat{x}_{k+1}'''(0))+C_{m-j}(\hat{x}_m'''(-\tau)-\hat{x}_m'(-\tau))=0,\quad j=[1:m-1],$$

$$C_0(\hat{x}_m'''(-\tau)-\hat{x}_m'(-\tau))+\sum_{k=1}^{m}A_{m-k}^{\top}\sum_{q=k}^{m}C_{q-k}(\hat{x}_q''(-\tau)-\hat{x}_q(-\tau))+\alpha^{-1}(\hat{x}_1(0)-\varphi_m(-\tau))=0.$$

In view of the definitions of the matrices C_i, $i=[0:m-1]$, we transform these boundary conditions to form (2.4) and (2.5). □

Lemma 1. *Let* $\det A_m\neq 0$. *Then,* $B=\|B_{jq}\|_{j,q=1}^{m}$ *is a symmetric positive definite matrix.*

Proof. Using formulas (2.6), which define the elements of the matrix B, we find that

$$B_{ji} = \sum_{k=j}^{m} A_{m+j-k}^{\top} A_{m+i-k} \quad \text{for} \quad 1 \le i < j \le m,$$

$$B_{jj} = \sum_{k=j}^{m} A_{m+j-k}^{\top} A_{m+j-k} \quad \text{for} \quad 1 \le j \le m,$$

$$B_{ji} = \sum_{k=i}^{m} A_{m+j-k}^{\top} A_{m+i-k} \quad \text{for} \quad 1 \le j < i \le m.$$

Therefore, $B_{ij}^{\top} = B_{ji}$ for $1 \le i < j \le m$ and $B_{jj}^{\top} = B_{jj}$ for $1 \le j \le m$.

Let us show that the quadratic form $\sum_{i,j=1}^{m} x_j^{\top} B_{ij} x_i$ is positive definite. We have

$$\sum_{i,j=1}^{m} x_j^{\top} B_{ij} x_i = \sum_{j=1}^{m} \sum_{k=j}^{m} x_j^{\top} A_{m+j-k}^{\top} \sum_{i=1}^{k} A_{m+i-k} x_i = \sum_{k=1}^{m} \sum_{j=1}^{k} x_j^{\top} A_{m+j-k}^{\top} \sum_{i=1}^{k} A_{m+i-k} x_i = \sum_{k=1}^{m} u_k^{\top} u_k,$$

$$u_k = \sum_{i=1}^{k} A_{m+i-k} x_i, \quad k = [1:m].$$

If there exists $u_k \ne 0$, $k = [1:m]$, then $x_k \ne 0$. Indeed, choosing the smallest of these k, we have $x_1 = A_m^{-1} u_1 \ne 0$ for $k = 1$; $u_j = 0$ and $x_j = 0$, $1 \le j < k$, and $x_k = A_m^{-1} u_k \ne 0$ for $1 < k \le m$. Consequently, the quadratic form is positive definite. □

3. ASYMPTOTICS OF REGULARIZED SOLUTIONS

For finding an asymptotic representation for the general solution of system of equations (2.1), methods of asymptotic integration of singular ordinary differential equations [12, p. 48; 13, p. 223] and the results of paper [9] are used.

According to the lemma, eigenvalues of the matrix B are real and positive [14]. Then, by a nonsingular orthogonal transformation defined by the matrix T, the matrix B is reduced to the Jordan form $B = T^{\top} J T$, where $J = \operatorname{diag}(\lambda_1, \dots, \lambda_{nm})$; here, λ_k, $k = [1:nm]$, are the eigenvalues of the matrix B.

Assertion 2 [9, Lemma 2]**.** *Assume that* $\varphi \in W_2^2([-r,0], \mathbb{R}^n)$*,* $\det A_m \ne 0$*, and eigenvalues of the matrix* B *are simple. Then, the general solution of system of differential equations* (2.1) *is defined by the asymptotic formula*

$$\hat{x}(s,\varphi,\alpha) = T^{\top} \sum_{j=1}^{4} K_j(\alpha) \Big(\exp(J_j(\alpha)(s-s_j)) D_j + \alpha^{-1/4} J_j^{-1}(\alpha) \Big(-G_j(\alpha) f(s) + \int_{s_j}^{s} \exp(J_j(\alpha)(s-t)) G_j(\alpha) f'(t)\, dt \Big) \Big), \quad s \in [-\tau, 0], \tag{3.1}$$

where D_j *are arbitrary elements from* $\mathbb{C}^{nm}$;

$$J_j(\alpha) = \operatorname{diag}(\lambda_{1j}(\alpha), \dots, \lambda_{nmj}(\alpha)),$$

$$\lambda_{kj}(\alpha) = \alpha^{-1/4}(\lambda_k^{1/4} e_j + O(\alpha^{1/4}; k, j)), \quad k = [1 : nm]; \quad s_1 = s_2 = 0, \quad s_3 = s_4 = -\tau;$$

$$e = (1+i)/\sqrt{2}, \quad e_1 = \bar{e}, \quad e_2 = e, \quad e_3 = -\bar{e}, \quad e_4 = -e; \quad K_j(\alpha) = I_{nm} + \mathbb{O}(\alpha^{1/4}; j);$$

$$G_j(\alpha) = -1/4\, e_j J^{-3/4} + \mathbb{O}(\alpha^{3/4}; j), \quad j = [1 : 4].$$

Formula (3.1) is used for finding an asymptotic solution of boundary value problem (2.1)–(2.5). Formula (3.1) implies the following asymptotic formulas:

$$\begin{aligned}
&\hat{x}(-\tau, \varphi, \alpha) = B^{-1} T^\top f(-\tau) + \widetilde{D}_3 + \widetilde{D}_4 + \mathcal{O}(\alpha^{1/4}; \widetilde{D}) + \mathcal{O}(\alpha^{1/4}; \varphi),\\
&\hat{x}(0, \varphi, \alpha) = B^{-1} T^\top f(0) + \widetilde{D}_1 + \widetilde{D}_2 + \mathcal{O}(\alpha^{1/4}; \widetilde{D}) + \mathcal{O}(\alpha^{1/4}; \varphi),\\
&\alpha^{1/4} \hat{x}'(-\tau, \varphi, \alpha) = e_3 B^{1/4} \widetilde{D}_3 + e_4 B^{1/4} \widetilde{D}_4 + \mathcal{O}(\alpha^{1/4}; \widetilde{D}) + \mathcal{O}(\alpha^{1/4}; \varphi),\\
&\alpha^{1/4} \hat{x}'(0, \varphi, \alpha) = e_1 B^{1/4}; \widetilde{D}_1 + e_2 B^{1/4} \widetilde{D}_2 + \mathcal{O}(\alpha^{1/4}; \widetilde{D}) + \mathcal{O}(\alpha^{1/4}; \varphi),\\
&\alpha^{1/2} \hat{x}''(-\tau, \varphi, \alpha) = e_3^2 B^{1/2} \widetilde{D}_3 + e_4^2 B^{1/2} \widetilde{D}_4 + \mathcal{O}(\alpha^{1/4}; \widetilde{D}) + \mathcal{O}(\alpha^{1/4}; \varphi),\\
&\alpha^{1/2} \hat{x}''(0, \varphi, \alpha) = e_1^2 B^{1/2} \widetilde{D}_1 + e_2^2 B^{1/2} \widetilde{D}_2 + \mathcal{O}(\alpha^{1/4}; \widetilde{D}) + \mathcal{O}(\alpha^{1/4}; \varphi),\\
&\alpha^{3/4} \hat{x}'''(-\tau, \varphi, \alpha) = e_3^3 B^{3/4} \widetilde{D}_3 + e_4^3 B^{3/4} \widetilde{D}_4 + \mathcal{O}(\alpha^{1/4}; \widetilde{D}) + \mathcal{O}(\alpha^{1/4}; \varphi),\\
&\alpha^{3/4} \hat{x}'''(0, \varphi, \alpha) = e_1^3 B^{3/4} \widetilde{D}_1 + e_2^3 B^{3/4} \widetilde{D}_2 + \mathcal{O}(\alpha^{1/4}; \widetilde{D}) + \mathcal{O}(\alpha^{1/4}; \varphi),
\end{aligned} \tag{3.2}$$

where $\widetilde{D}_k = T^\top D_k$, $k = [1 : 4]$, and $\widetilde{D} = \|\widetilde{D}_k\|_{k=1}^4$. Here, $\mathcal{O}(\alpha^{1/4}; \cdot) \colon \mathbb{C}^{4nm} \to \mathbb{C}^n$ and $\mathcal{O}(\alpha^{1/4}; \cdot) \colon H^2 \to \mathbb{C}^n$ are linear continuous mappings.

In view of these formulas, we transform boundary conditions (2.3)–(2.5) to the form

$$\begin{gathered}
\alpha^{3/4} \sum_{k=1}^{m} C_{k-1} \hat{x}_k'''(0, \varphi, \alpha) + \mathcal{O}(\alpha^{1/4}; \widetilde{D}) = \mathcal{O}(\alpha^{1/4}; \varphi),\\
\alpha^{1/2}(\hat{x}_j''(-\tau, \varphi, \alpha) - \hat{x}_{j+1}''(0, \varphi, \alpha)) + \mathcal{O}(\alpha^{1/4}; \widetilde{D}) = \mathcal{O}(\alpha^{1/4}; \varphi), \quad j = [1 : m-1],\\
\alpha^{1/2} \hat{x}_m''(-\tau, \varphi, \alpha) + \mathcal{O}(\alpha^{1/4}; \widetilde{D}) = \mathcal{O}(\alpha^{1/4}; \varphi),\\
\alpha^{3/4}(\hat{x}_j'''(-\tau, \varphi, \alpha) - \hat{x}_{j+1}'''(0, \varphi, \alpha)) + \mathcal{O}(\alpha^{1/4}; \widetilde{D}) = \mathcal{O}(\alpha^{1/4}; \varphi), \quad j = [1 : m-1],\\
A_m^\top(\hat{x}_1(0) - \varphi_m(-\tau)) + \mathcal{O}(\alpha^{1/4}; \widetilde{D}) = \mathcal{O}(\alpha^{1/4}; \varphi).
\end{gathered} \tag{3.3}$$

Substituting the values of the functions $\hat{x}_k(\cdot, \varphi, \alpha)$, $k = [1 : m]$, and their derivatives defined by asymptotic formulas (3.2) into (3.3), we obtain a linear nonhomogeneous system of algebraic equations for finding $\widetilde{D}_1, \dots, \widetilde{D}_4$. The coefficients of this system depend continuously on α, and, for $\alpha = 0$, the system takes the form

$$\begin{gathered}
(E_2 + e_1^3 S_m C B^{3/4}) \widetilde{D}_1 + (E_2 + e_2^3 S_m C B^{3/4}) \widetilde{D}_2 - E_1(\widetilde{D}_3 + \widetilde{D}_4)\\
= B^{-1} T^\top f(-\tau) - E_2 B^{-1} T^\top f(0),\\
B^{-1/4} E_2 B^{1/4}(e_1 \widetilde{D}_1 + e_2 \widetilde{D}_2) - e_3 \widetilde{D}_3 - e_4 \widetilde{D}_4 = 0,\\
B^{-1/2} E_2 B^{1/2}(e_1^2 \widetilde{D}_1 + e_2^2 \widetilde{D}_2) - e_3^2 \widetilde{D}_3 - e_4^2 \widetilde{D}_4 = 0,\\
(S_m A_m^\top S_1 + e_1^3 E_2 B^{3/4}) \widetilde{D}_1 + (S_m A_m^\top S_1 + e_2^3 E_2 B^{3/4}) \widetilde{D}_2 - e_3^3 E_1 B^{3/4} \widetilde{D}_3 - e_4^3 E_1 B^{3/4} \widetilde{D}_4\\
= S_m A_m^\top (S_m \varphi_m(-\tau) - S_1 B^{-1} T^\top f(0)).
\end{gathered}$$

From the second and third equations of this system, we find that

$$\widetilde{D}_3^0(\varphi) = -\frac{1}{\sqrt{2}}(\bar{e}F_1 - eF_2)\widetilde{D}_1^0(\varphi) - \frac{1}{\sqrt{2}}(eF_1 - eF_2)\widetilde{D}_2^0(\varphi),$$
$$\widetilde{D}_4^0(\varphi) = -\frac{1}{\sqrt{2}}(\bar{e}F_1 + \bar{e}F_2)\widetilde{D}_1^0(\varphi) - \frac{1}{\sqrt{2}}(eF_1 - \bar{e}F_2)\widetilde{D}_2^0(\varphi),$$

where $F_1 = B^{-1/4}E_2B^{1/4}$, $F_2 = B^{-1/2}E_2B^{1/2}$, $E_2 = \begin{pmatrix} 0_{n(m-1)\times n} & I_{n(m-1)\times n(m-1)} \\ 0_{n\times n} & 0_{n\times n(m-1)} \end{pmatrix}$, and $\widetilde{D}_1^0(\varphi)$ and $\widetilde{D}_2^0(\varphi)$ are solutions of the linear nonhomogeneous system

$$\begin{aligned} &\Big(E_2 - eS_mCB^{3/4} + \frac{1}{\sqrt{2}}E_1(2\bar{e}F_1 - (e-\bar{e})F_2)\Big)\widetilde{D}_1 + \Big(E_2 - \bar{e}S_mCB^{3/4} \\ &\quad + \frac{1}{\sqrt{2}}E_1(2eF_1 - (e-\bar{e})F_2)\Big)\widetilde{D}_2 = B^{-1}T^\top f(-\tau) - E_2B^{-1}T^\top f(0), \\ &\Big(S_mA_m^\top S_1 - eE_2B^{3/4} - \frac{1}{\sqrt{2}}E_1((1-i)F_1 - 2iF_2)\Big)\widetilde{D}_1 + \Big(S_mA_m^\top S_1 - \bar{e}E_2B^{3/4} \\ &\quad + \frac{1}{\sqrt{2}}E_1B^{3/4}((1+i)F_1 + 2iF_2)\Big)\widetilde{D}_2 = S_mA_m^\top(S_m\varphi_m(-\tau) - S_1B^{-1}T^\top f(0)). \end{aligned} \tag{3.4}$$

Here,

$$E_1 = \begin{pmatrix} I_{n(m-1)\times n(m-1)} & 0_{n(m-1)\times n} \\ 0_{n\times n(m-1)} & 0_{n\times n} \end{pmatrix}, \quad S_1 = \begin{pmatrix} I_{n\times n} \\ 0_{n(m-1)\times n} \end{pmatrix},$$

$$S_m = \begin{pmatrix} 0_{n(m-1)\times n} \\ I_{n\times n} \end{pmatrix}, \quad C = (C_0, \ldots, C_{m-1}).$$

Theorem 3. *Assume that $\varphi \in W_2^2([-r,0],\mathbb{R}^n)$, $\det A_m \neq 0$, eigenvalues of the matrix B are simple, and the determinant of system (3.4) is nonzero. Then, a solution of boundary value problem (2.1)–(2.5) is defined by the asymptotic formula*

$$\hat{x}(s,\varphi,\alpha) = T^\top \sum_{j=1}^{4} \exp(J_j(\alpha)(s-s_j))T\widetilde{D}_j^0(\varphi) + J^{-1}f(s) + \mathcal{O}(\alpha^{1/4}; s, \varphi), \quad s \in [-\tau, 0],$$

where $J_j(\alpha)$ and s_j, $j = [1:4]$, are defined in Assertion 2.

Proof. The validity of the theorem follows from Assertion 2 and the asymptotic solution of linear nonhomogeneous system (3.3). □

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