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THE STRUCTURE OF THE CATEGORY OF PARABOLIC EQUATIONS. I[1]

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We define here the category of partial differential equations. Special cases of morphisms from an object (equation) are symmetries of the equation and reductions of the equation by a symmetry groups, but there are many other morphisms. We are mostly interested in a subcategory that arises from second order parabolic equations on arbitrary manifolds. We develop a special-purpose language for description and study of the internal structure of such subcategories.

Keywords: category of partial differential equations, factorization of differential equations; parabolic equation; reaction-diffusion equation; heat equation; symmetry group.

СТРУКТУРА КАТЕГОРИИ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ. I

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В статье даётся определение категории дифференциальных уравнений в частных производных. Специальными классами морфизмов из объекта (уравнения) являются симметрии уравнения и редукции уравнения по группе симметрий, однако имется и много других морфизмов. Особое внимание в статье уделяется подкатегории, возникающей из параболических уравнений второго порядка, задданных на произвольных многообразиях. Развивается специальный язык для описания и изучения внутренней структуры таких подкатегорий.

Ключевые слова: категория дифференциальных уравнений в частных производных; факторизация дифференциальных уравнений; параболические уравнения; уравнение реакции-диффузии; уравнение теплопроводности; группа симметрий.

Introduction

In this paper we define the category $\mathcal{PDE}$ of partial differential equations, develop a special-purpose language for description and study of its internal structure, and consider its full subcategory $\mathcal{PE}$ that arises from second order parabolic equations on arbitrary manifolds. In the next paper [8] we investigate the structure of $\mathcal{PE}$ in detail.

$$\begin{tikzcd} N \arrow[r, "F"] \arrow[d, "\pi"'] & N' \arrow[d, "\pi'"] \\ M \arrow[r] & M' \end{tikzcd}$$

Рис. 1

Let us define first the category $\mathcal{PDE}_0$, which is a full subcategory of $\mathcal{PDE}$.

Let $\pi\colon N \to M$ be a smooth fiber bundle, E be a subset of k-jet bundle $J^k(\pi)$. E could be considered as a k-th order partial differential equation for sections of π; namely, $s \in \Gamma\pi$ is a solution of E if k-th prolongation $j^k(s) \in \Gamma\left(J^k(\pi) \to M\right)$ is contained in E. Here $\Gamma\pi$ denotes the space of smooth sections of π.

Let $\pi\colon N \to M$, $\pi'\colon N' \to M'$ be smooth fiber bundles. Suppose $F\colon \pi \to \pi'$ is a smooth bundle morphism with the additional property that F is a diffeomorphism on the fibers (see Fig. 1). Then F induces the map $F^*\colon \Gamma\pi' \to \Gamma\pi$; denote by $\Gamma_F\pi$ its image. We say that a section of π is F-projectable if it is contained in $\Gamma_F\pi$. If F is surjective then F^* is injective, so it defines the map from $\Gamma_F\pi$ to $\Gamma\pi'$. If additionally F is submersive then it defines the map $F^k\colon J^k_F(\pi) \to J^k(\pi')$, where $J^k_F(\pi) = F^* J^k(\pi')$ is the bundle of k-jets of F-projectable sections of π (see Fig. 2).

$$\begin{tikzcd} J^k_F(\pi) \arrow[d] \arrow[dr, "F^k"] & \\ J^k(\pi) \arrow[d] & J^k(\pi') \arrow[d] \\ N \arrow[r, two heads, "F"] \arrow[d, "\pi"'] & N' \arrow[d, "\pi'"] \\ M \arrow[r, two heads] & M' \end{tikzcd}$$

Рис. 2

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Now we are ready to define $\mathcal{PDE}_0$. Its objects are pairs $\left(\pi\colon N \to M, E \subset J^k(\pi)\right)$, $k \in \mathbb{N}$, and morphisms from an object $\left(\pi\colon N \to M, E \subset J^k(\pi)\right)$ to an object $\left(\pi'\colon N' \to M', E' \subset J^k(\pi')\right)$ are smooth bundle morphisms $F\colon \pi \to \pi'$ satisfying the following conditions:

1. F defines surjective submersion $M \to M'$;
2. the diagram Fig. 1 is a pullback square in the category of smooth manifolds, that is for any $x \in M$ the map $\pi^{-1}(x) \to \pi'^{-1}(Fx)$ is a diffeomorphism;
3. $E \cap J^k_F(\pi) = \left(F^k\right)^{-1}(E')$.

If $F\colon (\pi, E) \to (\pi', E')$ is a morphism in $\mathcal{PDE}_0$ then F^* defines a bijection between the set of all solutions of E' and the set of all F-projectable solutions of E.

In Section 2 we define a bigger category $\mathcal{PDE}$, whose objects are pairs (N, E) where N is a smooth manifold and E is a subset of the bundle $J^k_m(N)$ of k-jets of m-dimensional submanifolds of N, and whose morphisms from (N, E) to (N', E') are maps $N \to N'$ sutisfying some analogue of conditions (1-3) above. By definition, the solutions of an equation E are smooth m-dimensional non-vertical integral manifolds of the Cartan distribution on $J^k_m(N)$, which are contained in E. Particularly, the set of solutions includes all m-dimensional submanifolds $L \subset N$ such that the k-th prolongation $j^k(L) \subset J^k_m(N)$ is contained in E. Any morphism $F\colon (N, E) \to (N', E')$ of $\mathcal{PDE}$ defines a bijection between the set of all solutions of E' and the set of all F-projectable solutions of E in the same manner as for $\mathcal{PDE}_0$.

The category $\mathcal{PDE}$ generalizes the notion of a symmetry group in two directions:

1. Automorphisms group of an object (N, E) in $\mathcal{PDE}$ is the symmetry group of the equation E.
2. For a symmetry group G of E the natural projection $N \to N/G$ defines the morphism $(N, E) \to (N/G, E/G)$ in $\mathcal{PDE}$. Here E/G is the equation describing G-invariant solutions of E.

Note that morphisms of $\mathcal{PDE}$ go beyond morphisms of this kind.

In Sections 2 and 6 we discuss the relations between our approach to the factorization of PDE and the other approaches.

Then we discuss the possibility of the introduction of a certain structure in $\mathcal{PDE}$ formed by a lattice of subcategories. These subcategories may be obtained by restricting to equations of specific kind (for example, elliptic, parabolic, hyperbolic, linear, quasilinear equations etc.) or to the morphisms of specific kind (for example, morphisms respecting the projection of N on a base manifold M as in $\mathcal{PDE}_0$) or both. When we interested in solutions of some equation it is useful to look for its quotient objects because every quotient object gives us a class of solutions of the original equation. It may happen that the position of an object in the lattice gives information on the morphisms from the object and/or on the kind of the simplest representatives of quotient objects. In Section 4 we develop a special-purpose language for description and study of such situations. We introduce a number of partial orders on the class of all subcategories of fixed category and depict these orders by various arrows (see Table 1 and Fig. 3). For instance, we say that a subcategory $\mathcal{C}_1$ is closed in a category $\mathcal{C}$ and depict $\mathcal{C} \bullet\!\!\longrightarrow \mathcal{C}_1$ if every morphism in $\mathcal{C}$ with source from $\mathcal{C}_1$ is a morphism in $\mathcal{C}_1$; we say that $\mathcal{C}_1$ is plentiful in $\mathcal{C}$ and depict $\mathcal{C} \dashrightarrow \mathcal{C}_1$ if for every $\mathbf{A} \in \mathrm{Ob}_{\mathcal{C}_1}$ and for every quotient object of $\mathbf{A}$ in $\mathcal{C}$ there exists a representative of this quotient object in $\mathcal{C}_1$; and so on.

We use this language in the next paper [8] for a detail study of the full subcategory $\mathcal{PE}$ of $\mathcal{PDE}$ that arises from second order parabolic equations posed on arbitrary manifolds, but we hope that our approach based on category theory may be useful for other types of PDE as well. An object of $\mathcal{PE}$ is an equation for an unknown function $u(t, x)$, $x \in X$ having the form $u_t = \sum_{i,j} b^{ij}(t, x, u)u_{ij} + \sum_{i,j} c^{ij}(t, x, u)u_i u_j + \sum_i b^i(t, x, u)u_i + q(t, x, u)$ in local coordinates $\left(x^i\right)$ on X, where X is a smooth manifold. We prove that every morphism in $\mathcal{PE}$ is of the form $(t, x, u) \mapsto (t'(t), x'\left(t, x\right), u'(t, x, u))$ (Theorem 1). Particularly, $\mathcal{PE}$ appears to be a subcategory of $\mathcal{PDE}_0$.

Using of the structure of $\mathcal{PE}$ developed in [8] is illustrated there on the example of the reaction-diffusion equation

$$u_t = a(u)\left(\Delta u + \eta \nabla u\right) + q(x, u), \quad x \in X, t \in \mathbb{R}, \tag{0.1}$$

posed on a Riemannian manifold X equipped with a vector field η. There are two exceptional cases: $a(u) = e^{\lambda u}H(u)$ and $a(u) = (u-u_0)^\lambda H(\ln(u-u_0))$, where $H(\cdot)$ is a periodic function; in these cases there are more morphisms then in a regular case. If only function $a(u)$ does not belong to one of these two exceptional classes then every morphism from equation (0.1) may be transformed by an isomorphism (i.e. by a global change of variables) of the quotient equation to a "canonical" morphism of very simple kind so that the "canonical" quotient equation has the same form as (0.1) with the same function $a(u)$ but is posed on another Riemannian manifold X', $\dim X' \le \dim X$.

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1. The "small" category $\mathcal{PDE}_0$ of partial differential equations

Let M, K be smooth manifolds. A system E of k-th order partial differential equations for a function $u\colon M \to K$ is given as a system of equations $\Phi^l(x, u, \dots, u^{(k)}) = 0$ involving x, u and the derivatives of u with respect to x up to order k, where $x = (x^1, \dots, x^m)$ are local coordinates on M and $u = (u^1, \dots, u^j)$ are local coordinates on K. Further we will use the words "partial differential equation", "PDE" or "equation" instead of "a partial differential equation or a system of partial differential equations" for short.

Recall some things about jets and related notions. The k-jet of a smooth function $u\colon M \to K$ at a point $x \in M$ is the equivalence class of smooth functions $M \to K$ whose value and partial derivatives up to k-th order at x coincide with the ones of u. All k-jets of all smooth functions $M \to K$ form the smooth manifold $J^k(M,K)$, and the natural projection $\pi^k\colon J^k(M,K) \to J^0(M,K) = M \times K$ defines a smooth vector bundle over $M \times K$, which is called k-jet bundle. For every function $u\colon M \to K$ its k-th prolongation $j^k(u)\colon M \to J^k(M,K)$ is naturally defined. k-th order PDE for functions acting from M to K can be considered as a subset E of $J^k(M,K)$; solutions of E are functions $u\colon M \to K$ such that the image of $j^k(u)$ is contained in E.

In more general situation we have a smooth fiber bundle $\pi\colon N \to M$ instead of a projection $M \times K \to M$, and sections $s\colon M \to N$ instead of functions $u\colon M \to K$. Denote by $\Gamma\pi$ the space of smooth sections of π; recall that a section of π is a map $s\colon M \to N$ such that $\pi \circ s$ is the identity. Definitions of the k-jet bundle $\pi^k\colon J^k(\pi) \to J^0(\pi) = N$ and of the k-th prolongation $j^k\colon \Gamma\pi \to \Gamma\left(\pi \circ \pi^k\colon J^k(\pi) \to M\right)$ are the same as ones for functions. Let E be a subset of $J^k(\pi)$; then E can be considered as a k-th order partial differential equation for sections of π, that is $s \in \Gamma\pi$ is a solution of E if the image of $j^k(s)$ is contained in E.

Let $\pi\colon N \to M$, $\pi'\colon N' \to M'$ be smooth fiber bundles. Let $F\colon \pi \to \pi'$ be a smooth bundle morphism with the additional property that F is a diffeomorphism on the fibers (see Fig. 1). F induces the map $F^*\colon \Gamma\pi' \to \Gamma\pi$; denote by $\Gamma_F\pi$ its image. We say that a section of π is *F-projectable* if it is contained in $\Gamma_F\pi$. If F is surjective then F^* is injective, so it defines the map $F_\#\colon \Gamma_F\pi \to \Gamma\pi'$. If additionally F is submersive then it defines the map $F^k\colon J^k_F(\pi) \to J^k(\pi')$, where $J^k_F(\pi) = F^*J^k(\pi')$ is the bundle of k-jets of F-projectable sections of π (see Fig. 2). Recall that a map F is called a submersion if $\mathrm{d}F\colon T_xN \to T_{F(x)}N'$ is surjective at each point $x \in N$.

D e f i n i t i o n 1. Let $\pi\colon N \to M$, $\pi'\colon N' \to M'$ be smooth fiber bundles, E be a subset of $J^k(\pi)$, $F\colon \pi \to \pi'$ be a smooth bundle morphism. We say that F *is admitted by* E if the following conditions are satisfied:

1. F is a surjective submersion;
2. the diagram Fig. 1 is a pullback square in the category of smooth manifolds, that is for any $x \in M$ the map $\pi^{-1}(x) \to \pi'^{-1}(Fx)$ is a diffeomorphism;
3. $E \cap J^k_F(\pi) = \left(F^k\right)^{-1}(E')$ fore some subset E' of $J^k(\pi')$.

If this is the case then we say that E' is *F-projection* of E.

It turns out that the language of category theory is very convenient for our study of PDE. Recall that a category $\mathcal{C}$ consists of a collection of *objects* $\mathrm{Ob}_{\mathcal{C}}$, a collection of *morphisms* (or *arrows*) $\mathrm{Hom}_{\mathcal{C}}$ and four operations. The first two operations associate with each morphism F of $\mathcal{C}$ its *source* and its *target*, both of which are objects of $\mathcal{C}$. The remaining two operations are an operation that associates with each object $\mathbf{C}$ of $\mathcal{C}$ an *identity morphism* $\mathrm{id}_{\mathbf{C}} \in \mathrm{Hom}_{\mathcal{C}}$ and an operation of *composition* that associates to any pair (F, G) of morphisms of $\mathcal{C}$ such that the source of F coincides with the target of G another morphism $F \circ G$, their composite. These operations should satisfy some natural axioms [3].

D e f i n i t i o n 2. $\mathcal{PDE}_0$ is the category whose objects are pairs $\big(\pi\colon N \to M, E \subset J^k(\pi)\big)$ with π being a smooth viber bundle, $k \in \mathbb{N}$, and morphisms from an object $\big(\pi\colon N \to M, E \subset J^k(\pi)\big)$ to an object $\big(\pi'\colon N' \to M', E' \subset J^k(\pi')\big)$ are smooth bundle morphisms $F\colon \pi \to \pi'$ admitted by E such that E' is the F-projection of E.

If $F\colon (\pi, E) \to (\pi', E')$ is a morphism in $\mathcal{PDE}_0$ then F^* defines a bijection between the set of all solutions of E' and the set of all F-projectable solutions of E.

2. The category $\mathcal{PDE}$ of partial differential equations

In this section we define the category $\mathcal{PDE}$ of partial differential equations, whose objects are pairs (N, E) such that N is a smooth manifold and E is a subset of the bundle $J^k_m(N)$ of k-jets of m-dimensional submanifolds of N.

Let N be a C^r-smooth manifold, $0 < m < \dim N$. The jet bundle $\pi^k\colon J^k_m(N) \to N$ is a fiber bundle with the fiber $J^k_m(N)\big|_x$ over $x \in N$, where $J^k_m(N)\big|_x$ is the set of equivalence classes of smooth m-dimensional submanifolds L of N passing through x under the equivalence relation of k-th order contact in x.

The k-jet of a k-smooth m-dimensional submanifold L over $x \in L$ is the equivalence class from $J^k_m(N)\big|_x$ determined by L. Thus we have the prolongation map $j^k\colon L \to J^k_m(N)$ taking each point $x \in L$ to the k-jet of L over x (so it is the section of the fiber bundle $J^k_m(N)$ restricted to $L \subset N$). For every $k > l \geq 0$ the natural projection $\pi^{k,l}\colon J^k_m(N) \to J^l_m(N)$ maps the k-jet of L to the l-jet of L over x for every m-dimensional submanifold L of N and every $x \in L$.

For a submanifold L of N the differential of the prolongation map $j^k\colon L \to J^k_m(N)$ takes the tangent bundle TL to the tangent bundle $TJ^k_m(N)$. The closure of the union of the images of TL in $TJ^k_m(N)$ when L runs over all m-dimensional submanifolds of N is the vector subbundle of $TJ^k_m(N)$; it is called the Cartan distribution on $J^k_m(N)$.

Let E be a submanifold of $J^k(\pi)$, $\pi\colon N \to M$, $m = \dim M$. The graph of a section is an m-dimensional submanifold of N, so $J^k(\pi)$ is an open subspace of $J^k_m(N)$ and E could be considered as a submanifold of $J^k_m(N)$. The extended version of E is defined as the closure of E in $J^k_m(N)$ [4]. Since we don't plan to consider infinitesimal properties of E in contrast to the Lie group analysis of PDE, we would consider any subset E of $J^k_m(N)$ as a partial differential equations. By definition, *solutions* of such an equation are smooth m-dimensional non-vertical integral manifolds of the Cartan distribution on $J^k_m(N)$ that are contained in E. Note that for any m-dimensional submanifold L of N its prolongation $j^k(L)$ is a non-vertical integral manifold of the Cartan distribution on $J^k_m(N)$. Therefore, if $E \subset J^k_m(N)$ is obtained from a traditional PDE as it was described above, and if L is the graph of a section u of π, then L is a solution of E in the above sense if and only if u is a solution of the corresponding traditional PDE in the traditional sense. In addition there is allowed the possibility of both multi-valued solutions and solutions with infinite derivatives (see [4] for the details). Wherever we write concrete equation in the traditional form below we mean the extended version of this equations, that is the closure of the corresponding set in $J^k_m(N)$.

Now let us introduce some auxiliary notations.

Let $F\colon N \to N'$ be a map. We say that $L \subset N$ is *F-projected* if $L = F^{-1}(F(L))$. Note that if F is a surjective submersion and L is an F-projectable submanifold of N then $L' = F(L)$ is a submanifold of N'.

Let N, N' be C^r-smooth manifolds, $0 < m < \dim N$. Let $F\colon N \to N'$ be a surjective submersion of smoothness class C^s, $k \le s \le r$.

D e f i n i t i o n 3. *F-projectable jet bundle* $J^k_{m,F}(N)$ is the submanifold of $J^k_m(N)$ consisting of k-jets of all m-dimensional F-projectable submanifolds of N.

We write $J^k_F(N)$ instead of $J^k_{m,F}(N)$ if the value of m is clear from context.

There is natural isomorphism between the bundles $J^k_{m,F}(N)$ and $F^*J^k_{m'}(N')$ over N, where $F^*J^k_{m'}(N') = J^k_{m'}(N') \times_{N'} N$ is the pullback of $J^k_{m'}(N')$ by F, $\dim N - m = \dim N' - m'$. Therefore we can lift F to the map $F^k\colon J^k_{m,F}(N) \to J^k_{m'}(N')$ by the following natural way:
1) Suppose $\vartheta \in J^k_{m,F}(N)$. Take an arbitrary F-projectable submanifold L of N such that the k-th prolongation of L pass through ϑ (that is the k-jet of L over $\pi^k(\vartheta)$) is ϑ.
2) Assign to ϑ the point $\vartheta' \in J^k_{m'}(N')$, where ϑ' is the k-jet of the submanifold $L' = F(L) \subset N'$ over $F \circ \pi^k(\vartheta)$.

D e f i n i t i o n 4. Let $E \subset J^k_m(N)$. Let $F\colon N \to N'$ be a smooth surjective submersion. We say that F *is admitted by* E if the intersection $E \cap J^k_{m,F}(N)$ is F^k-projectable subset of $J^k_{m,F}(N)$ (see the left diagram on Fig. 3). Equivalently, $E \cap J^k_{m,F}(N)$ is the pre-image $\left(F^k\right)^{-1}(E')$ of some $E' \subset J^k_{m'}(N')$; we say that E' is *F-projection* of E.

$$
\begin{tikzcd}
E \arrow[d, tail] & E \cap J^k_{m,F}(N) \arrow[l, tail] \arrow[r, two heads] \arrow[d] & E' \arrow[d, tail] \\
J^k_m(N) \arrow[d, two heads, "\pi^k"'] & J^k_{m,F}(N) \arrow[l, tail] \arrow[r, two heads, "F^k"] \arrow[d, two heads] & J^k_{m'}(N') \arrow[d, two heads, "\pi'^k"'] \\
N \arrow[r, equal] & N \arrow[r, two heads, "F"'] & N'
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[d, tail] & E \cap \Delta \arrow[l, tail] \arrow[r, two heads] \arrow[d] & E' \arrow[d, tail] \\
J^k_m(N) \arrow[d, two heads, "\pi^k"'] & \Delta \arrow[l, tail] \arrow[r, two heads, "\tilde F"] \arrow[d, two heads] & J^{k'}_{m'}(N') \arrow[d, two heads, "\pi'^{k'}"'] \\
N \arrow[r, equal] & N & N'
\end{tikzcd}
$$

Рис. 3. Morphisms of $\mathcal{PDE}$ (left diagram) and morphisms of $\mathcal{PDE}_{\mathrm{ext}}$ (right diagram)

D e f i n i t i o n 5. The *category of partial differential equations $\mathcal{PDE}$* is defined as follows:

- objects of $\mathcal{PDE}$ are pairs (N, E), where N is a smooth manifold, E is a subset of $J^k_m(N)$ for some integer $k, m \ge 1$;
- morphisms of $\mathcal{PDE}$ with a source $\mathbf{A} = (N, E)$ are all surjective submersions $F\colon N \to N'$ admitted by E; target of such morphism is (N', E') where E' is F-projection of E;
- the identity morphism from $\mathbf{A}$ is the identity mapping of N, and the composition of morphisms is the composition of appropriate maps.

If $F\colon (N, E) \to (N', E')$ is a morphism in $\mathcal{PDE}$ then F^* defines a bijection between the set of all solutions of E' and the set of all F^k-projectable solutions of E.

In [5] we defined the following notion of a map admitted by a pair of equations: a map $F\colon N \to N'$ is admitted by an ordered pair of equations $(\mathbf{A}, \mathbf{A}')$, $\mathbf{A} = (N, E)$, $\mathbf{A}' = (N', E')$ if for any $L' \subset N'$ the following two conditions are equivalent:

- L' is the graph of a solution of E',
- $F^{-1}(L')$ is the graph of a solution of E.

However, we are not happy with this definition; in particular, because it deals only with global solutions of E. Therefore we now formulated the notion of a map admitted by an equation in terms of (locally defined) jet bundles.

R e m a r k 1. Let $\mathbf{A} = (N, E)$ be an object of $\mathcal{PDE}$. Then its automorphism group $\mathrm{Aut}(\mathbf{A})$ is the symmetry group for the equation E.

R e m a r k 2. Suppose G is a subgroup of the symmetry group of E such that N/G is a smooth manifold. Then the natural projection $N \to N/G$ defines the morphism $(N, E) \to (N/G, E/G)$ in $\mathcal{PDE}$. Here E/G is the equation describing G-invariant solutions of E.

Therefore, reduction of E by subgroups of $\mathrm{Aut}(\mathbf{A})$ defines a part of nontrivial morphisms from $\mathbf{A}$. But the class of all morphisms from $\mathbf{A}$ is significantly richer than the class of morphisms arising from reduction by subgroups of $\mathrm{Aut}(\mathbf{A})$. Let $\mathrm{Sol}(\mathbf{A})$ be the set of all solutions of $\mathbf{A}$, that is of all smooth m-dimensional non-vertical integral manifolds of the Cartan distribution on $J^k_m(N)$ that are contained in E. In general, the subset $F^*(\mathrm{Sol}(\mathbf{A}')) = \{F^{-1}(L') \colon L' \in \mathrm{Sol}(\mathbf{A}')\} \subseteq \mathrm{Sol}(\mathbf{A})$ of solutions of $\mathbf{A}$ arising from a morphism $F\colon \mathbf{A} \to \mathbf{A}'$ can *not* be represented as a set of solutions that are invariant under some subgroup of $\mathrm{Aut}(\mathbf{A})$. In particular, $F^*(\mathrm{Sol}(\mathbf{A}'))$ can be the set of G-invariant solutions, where G is a transformation group that is not necessarily a symmetry group of E. Moreover, for a morphism $F\colon \mathbf{A} \to \mathbf{A}'$ it may occur that for every nontrivial diffeomorphism g of N there is an element in $F^*(\mathrm{Sol}(\mathbf{A}'))$ that is not g-invariant. More detailed discussion is given in Section 6; see also [6], [7].

Our approach is conceptually close to the approach developed in [1] that deals with control systems. If we set aside the control part and look at this approach relative to ordinary differential equations, then we get the category of ordinary differential equations, whose objects are ODE systems of the form $\dot{x} = \xi$, $x \in X$, where X is a manifold equipped with a vector field ξ, and morphism from a system $\mathbf{A}$ to a system $\mathbf{A}'$ is a smooth map F from the phase space X of $\mathbf{A}$ to the phase space X' of $\mathbf{A}'$ that projected ξ to ξ'. In other words, F is a morphism if it transforms solutions (phase trajectories) of $\mathbf{A}$ to solutions of $\mathbf{A}'$: $F_*(\mathrm{Sol}(\mathbf{A})) = \mathrm{Sol}(\mathbf{A}')$.

By contrast, we deal with pullbacks of the solutions of the quotient equation $\mathbf{A}'$ to the solutions of the original equation $\mathbf{A}$. In our approach the number of dependent variables in the reduced PDE remains the same, while the number of independent variables is not increased. Thus in the approach proposed the quotient object notion is an analogue of the sub-object notion (in terminology of [1]) with respect to the information about the solutions of the given equation; however, it is similar to the quotient object notion with respect to interrelations between the given and reduced equations.

Note also that described above category of ODE from [1] is isomorphic to certain subcategory of $\mathcal{PDE}$. Namely, let us consider the following subcategory $\mathcal{PDE}_1$ of $\mathcal{PDE}$:

- objects of $\mathcal{PDE}_1$ are pairs (N, E), where $N = X \times \mathbb{R}$, E is a first order linear PDE of the form $L_\xi u = 1$ for unknown function $u\colon X \to \mathbb{R}$, $\xi \in TX$;
- morphisms of $\mathcal{PDE}_1$ are morphisms of $\mathcal{PDE}$ of the form $(x, u) \mapsto (x'(x), u)$.

One can easily see that the category of ODE from [1] is isomorphic to $\mathcal{PDE}_1$: the object $L_\xi u = 1$ corresponds to the object $\dot{x} = \xi$, and the morphism $(x, u) \mapsto (x'(x), u)$ corresponds to the morphism $x \mapsto x'(x)$.

The category of differential equations was also defined in [2] in a different way: objects are infinite-dimensional manifolds endowed with integrable finite-dimensional distribution (particularly, infinite prolongations of differential equations), and morphisms are smooth maps such that image of the distribution is contained in the distribution on the image, similarly to morphisms in [1]. Thus, the category of differential equations defined in [2] is quite different from the category $\mathcal{PDE}$ defined here; one should keep it in mind in order to avoid confusion. The factorization of PDE $\mathbf{A}$ by a symmetry group described in [2] is PDE $\mathbf{A}'$ on the quotient space describing images of all solutions of $\mathbf{A}$ at the projection to the quotient space: $F_*(\mathrm{Sol}(\mathbf{A})) = \mathrm{Sol}(\mathbf{A}')$. In that approach every factorization of $\mathbf{A}$ provides a part of the information about all the solutions of $\mathbf{A}$. In our approach factorization of $\mathbf{A}$ is such an equation $\mathbf{A}'$ that the pullbacks of its solutions are solutions of $\mathbf{A}$: $F^*(\mathrm{Sol}(\mathbf{A}')) \subseteq \mathrm{Sol}(\mathbf{A})$; so that from every factorization we obtain the full information about a certain set of the solutions of the given equation.

The following two propositions are simple corollaries of our definitions.

Proposition 1. *All morphisms in $\mathcal{PDE}$ are epimorphisms.*

Proposition 2. *Suppose (N, E), (N', E'), (N'', E'') are objects of $\mathcal{PDE}$, $F\colon N \to N'$ is a morphism from (N, E) to (N', E') in $\mathcal{PDE}$, $G\colon N' \to N''$ is surjective submersion. Then the following two conditions are equivalent (see Fig. 4):*

- *G is a morphism from (N', E') to (N'', E''),*
- *GF is a morphism from (N, E) to (N'', E'').*

$$
\begin{tikzcd}
E \cap J^k_{GF}(N) \arrow[d, tail] \arrow[dr, two heads] \arrow[ddrr, bend left, "(GF)^k"] & & \\
J^k_{GF}(N) \arrow[d, tail] \arrow[dr, two heads] & E' \cap J^k_G(N') \arrow[d, tail] \arrow[dr, two heads] & \\
J^k_F(N) \arrow[d, tail] \arrow[dr, two heads, "F^k"] & J^k_G(N') \arrow[d, tail] \arrow[dr, two heads, "G^k"] & E'' \arrow[d, tail] \\
J^k(N) \arrow[d, "\pi_k"'] & J^k(N') \arrow[d, "\pi'_k"'] & J^k(N'') \arrow[d, "\pi''_k"'] \\
N \arrow[r, two heads, "F"'] & N' \arrow[r, two heads, "G"'] & N''
\end{tikzcd}
$$

Рис. 4. Diagram for Proposition 2

3. The extended category $\mathcal{PDE}_{\mathrm{ext}}$ of partial differential equations

Note that the Cartan distribution $C^k(N)$ on $J^k_m(N)$ restricted to $J^k_{m,F}(N)$ coincides with the lifting $\left(F^k\right)^* C^k_{m'}(N')$ of the Cartan distribution on $J^k_{m'}(N')$, $m' = m - \dim N + \dim N'$. Taking this into account and using the analogy with higher symmetry group, we replace $J^k_{m,F}(N)$ to arbitrary submanifold Δ of $J^k_m(N)$. Thus we obtain the category $\mathcal{PDE}_{\mathrm{ext}}$ with the same objects as $\mathcal{PDE}$ and extended set of morphisms involving transformations of jets. (This category will not be used in the rest of the paper.)

D e f i n i t i o n 6. An *extended category of partial differential equations $\mathcal{PDE}_{\mathrm{ext}}$* is defined as follows:

- objects of $\mathcal{PDE}_{\mathrm{ext}}$ are pairs (N, E), where N is a smooth manifold, E is a subset of $J^k_m(N)$ for some integer $k, m \geq 1$;
- morphisms of $\mathcal{PDE}_{\mathrm{ext}}$ from $\mathbf{A} = (N, E \subset J^k_m(N))$ to $\mathbf{A}' = (N', E' \subset J^{k'}_{m'}(N'))$ are all pairs $\left(\Delta, \tilde{F}\right)$ such that Δ is a smooth submanifold of $J^k_m(N)$, $\tilde{F}\colon \Delta \to J^{k'}_{m'}(N')$ is a surjective submersion, the Cartan distribution on $J^k_m(N)$ restricted to Δ coincides with the lifting $\tilde{F}^* C^{k'}_{m'}(N')$ of the Cartan distribution on $J^{k'}_{m'}(N')$, and $E \cap \Delta = \tilde{F}^{-1}(E')$ (see right diagram on Fig. 3);

- the identity morphism from $\mathbf{A}$ is $\Big(\Delta = J^k_m(N), \tilde{F} = \mathrm{id}_N\Big)$;
- composition of $\Big(\Delta \subset J^k_m(N), \tilde{F}\colon \Delta \to J^{k'}_{m'}(N')\Big)$ and $\Big(\Delta' \subset J'^{k'}_m(N'), \tilde{F}'\colon \Delta' \to J^{k''}_{m''}(N'')\Big)$ is $\Big(\tilde{F}^{-1}(\Delta'), \tilde{F}' \circ \tilde{F}\Big)$.

For each integral manifold of the Cartan distribution on E' its preimage is an integral manifold of the Cartan distribution on E, so for each solution of E' its pullback is some solution of E.

$\mathcal{PDE}$ embeds to $\mathcal{PDE}_{\mathrm{ext}}$ by the following natural way: to the morphisms $F\colon N \to N'$ of $\mathcal{PDE}$ from the equation of k-th order we assign the morphisms $\Big(\Delta, \tilde{F}\Big)$ of $\mathcal{PDE}_{\mathrm{ext}}$ such that $\Delta = J^k_{m,F}(N)$, $\tilde{F} = F^k$.

4. Usage of subcategories

We start with review of some basic definitions of category theory [3]. Given a category $\mathcal{C}$ and an object $\mathbf{A}$ of $\mathcal{C}$, one may construct the category $(\mathbf{A} \downarrow \mathcal{C})$ of objects under $\mathbf{A}$ (this is the particular case of the comma category): objects of $(\mathbf{A} \downarrow \mathcal{C})$ are morphisms of $\mathcal{C}$ with source $\mathbf{A}$, and morphisms of $(\mathbf{A} \downarrow \mathcal{C})$ from one such object $F\colon \mathbf{A} \to \mathbf{B}$ to another $F'\colon \mathbf{A} \to \mathbf{B}'$ are morphisms $G\colon \mathbf{B} \to \mathbf{B}'$ of $\mathcal{C}$ such that $F' = G \circ F$.

Suppose $\mathcal{C}$ is a subcategory of $\mathcal{PDE}$, $\mathbf{A}$ is an object of $\mathcal{C}$. Then the category $(\mathbf{A} \downarrow \mathcal{C})$ of objects under $\mathbf{A}$ describes collection of quotient equations for $\mathbf{A}$ and their interconnection in the framework of $\mathcal{C}$.

To each morphism $F\colon \mathbf{A} \to \mathbf{B}$ with source $\mathbf{A}$ (that is to each object of the comma category $(\mathbf{A} \downarrow \mathcal{C})$) assign the set $F^*(\mathrm{Sol}(\mathbf{B})) \subseteq \mathrm{Sol}(\mathbf{A})$ of such solutions of $\mathbf{A}$ that “projected” onto underying space of $\mathbf{B}$ (space of dependent and independent variables). We can identify such morphisms that generated the same sets of solutions of $\mathbf{A}$, that is identify isomorphic objects of the comma category $(\mathbf{A} \downarrow \mathcal{C})$.

Describe the situation more explicitly. An equivalence class of epimorphisms with source $\mathbf{A}$ is called a quotient object of $\mathbf{A}$, where two epimorphisms $F\colon \mathbf{A} \to \mathbf{B}$ and $F'\colon \mathbf{A} \to \mathbf{B}'$ are equivalent if $F' = I \circ F$ for some isomorphism $I\colon \mathbf{B} \to \mathbf{B}'$ [3]. If $F\colon \mathbf{A} \to \mathbf{B}$ and $F'\colon \mathbf{A} \to \mathbf{B}'$ are equivalent, then they lead to the same subsets of the solutions of $\mathbf{A}$: $F^*(\mathrm{Sol}(\mathbf{B})) = F'^*(\mathrm{Sol}(\mathbf{B}'))$. So if we interested only in the sets of the solutions of $\mathbf{A}$, then all representatives of the same quotient object have the same rights.

Therefore, the following problems naturally arise:

- to study all morphisms with given source,
- to choose a "simplest"representative from every equivalence class, or to choose representative with the simplest target (that is the simplest quotient equation).

In order to deal with these problems we develop in this paper a special-purpose language.

Let us introduce a number of partial orders on the class of all categories to describe arising situations. First of all, we define a few types of subcategories.

D e f i n i t i o n 7. Suppose $\mathcal{C}$ is a category, $\mathcal{C}_1$ is a subcategory of $\mathcal{C}$.

- $\mathcal{C}_1$ is called a *wide* subcategory of $\mathcal{C}$ if all objects of $\mathcal{C}$ are objects of $\mathcal{C}_1$.
- $\mathcal{C}_1$ is called a *full* subcategory of $\mathcal{C}$ if every morphism in $\mathcal{C}$ with source and target from $\mathcal{C}_1$ is a morphism in $\mathcal{C}_1$.
- We say that $\mathcal{C}_1$ is *full under isomorphisms in* $\mathcal{C}$ if every isomorphism in $\mathcal{C}$ with source and target from $\mathcal{C}_1$ is an isomorphism in $\mathcal{C}_1$.
- We say that $\mathcal{C}_1$ is *closed in* $\mathcal{C}$ if every morphism in $\mathcal{C}$ with source from $\mathcal{C}_1$ is a morphism in $\mathcal{C}_1$. (Note that every subcategory that is closed in $\mathcal{C}$ is full in $\mathcal{C}$.)
- We say that $\mathcal{C}_1$ is *closed under isomorphisms in* $\mathcal{C}$ if every isomorphism in $\mathcal{C}$ with source from $\mathcal{C}_1$ is an isomorphism in $\mathcal{C}_1$.

- We say that $\mathcal{C}_1$ is *dense in* $\mathcal{C}$ if every object of $\mathcal{C}$ is isomorphic in $\mathcal{C}$ to an object of $\mathcal{C}_1$.
- We say that $\mathcal{C}_1$ is *plentiful in* $\mathcal{C}$ if for every morphism $F\colon \mathbf{A} \to \mathbf{B}$ in $\mathcal{C}$, $\mathbf{A} \in \mathrm{Ob}_{\mathcal{C}_1}$, there exists an isomorphism $I\colon \mathbf{B} \to \mathbf{C}$ in $\mathcal{C}$ such that $I \circ F \in \mathrm{Hom}_{\mathcal{C}_1}$ (in other words, for every quotient object of $\mathbf{A}$ in $\mathcal{C}$ there exists a representative of this quotient object in $\mathcal{C}_1$). Such morphism $I \circ F$ we call $\mathcal{C}_1$**-canonical** for F.
- We say that $\mathcal{C}_1$ is *fully dense (fully plentiful) in* $\mathcal{C}$ if $\mathcal{C}_1$ is a full subcategory of $\mathcal{C}$ and $\mathcal{C}_1$ is dense (plentiful) in $\mathcal{C}$.

The first two parts of this definition are standard notions of category theory, while the notions of the other parts are introduced here for the sake of description of the structure of $\mathcal{PDE}$.

R e m a r k 3. Using the notion of “the category of objects under $\mathbf{A}$”, we can define the notions of closed subcategory and plentiful subcategory by the following way:

- $\mathcal{C}_1$ is closed in $\mathcal{C}$ if for each $\mathbf{A} \in \mathrm{Ob}_{\mathcal{C}_1}$ the category $(\mathbf{A} \downarrow \mathcal{C}_1)$ is wide in $(\mathbf{A} \downarrow \mathcal{C})$.
- $\mathcal{C}_1$ is plentiful in $\mathcal{C}$ if for each $\mathbf{A} \in \mathrm{Ob}_{\mathcal{C}_1}$ the category $(\mathbf{A} \downarrow \mathcal{C}_1)$ is dense in $(\mathbf{A} \downarrow \mathcal{C})$.

R e m a r k 4. $\mathcal{C}_1$ is fully dense in $\mathcal{C}$ if and only if the embedding functor $\mathcal{C}_1 \to \mathcal{C}$ defines an equivalence of these categories.

Choose some category $\mathcal{U}$, which is big enough to contain all needful for us categories as it's subcategories. For our purposes $\mathcal{U} = \mathcal{PDE}$ is sufficient.

Define the category $\mathcal{U}_{\geq}$, whose objects are subcategories $\mathcal{C}$ of $\mathcal{U}$, and a collection $\mathrm{Hom}_{\mathcal{U}_{\geq}}(\mathcal{C}_1, \mathcal{C}_2)$ of morphisms from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a one-element set if $\mathcal{C}_2$ is subcategory of $\mathcal{C}_1$ and empty otherwise, so an arrow from $\mathcal{C}_1$ to $\mathcal{C}_2$ in $\mathcal{U}_{\geq}$ means that $\mathcal{C}_2$ is the subcategory of $\mathcal{C}_1$. Let $\mathcal{U}_{=}$ be the discrete wide subcategory of $\mathcal{U}_{\geq}$, that is objects of $\mathcal{U}_{=}$ are all subcategories $\mathcal{C}$ of $\mathcal{U}$, and the only morphisms are identities, so $\mathcal{C}_1$ and $\mathcal{C}_2$ are connected by arrow in $\mathcal{U}_{=}$ only if $\mathcal{C}_1 = \mathcal{C}_2$.

D e f i n i t i o n 8. Suppose $\mathcal{C}_1$, $\mathcal{C}_2$ are subcategories of $\mathcal{U}$. A subcategory of $\mathcal{U}$, whose objects are objects of $\mathcal{C}_1$ and $\mathcal{C}_2$ simultaneously, and whose morphisms are morphisms of $\mathcal{C}_1$ and $\mathcal{C}_2$ simultaneously, is called an *intersection* of $\mathcal{C}_1$ and $\mathcal{C}_2$ and is denoted by $\mathcal{C}_1 \cap \mathcal{C}_2$. In other words, $\mathcal{C}_1 \cap \mathcal{C}_2$ is the fibered sum of $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathcal{U}_{\geq}$.

The following proposition is obvious:

Proposition 3. *Suppose $\mathcal{C}_1$ is closed in $\mathcal{C}$, and $\mathcal{C}_2$ is (full/closed/dense/plentiful) subcategory of $\mathcal{C}$; then $\mathcal{C}_1 \cap \mathcal{C}_2$ is closed in $\mathcal{C}_2$ and is (full/closed/dense/plentiful) subcategory of $\mathcal{C}_1$.*

Now we introduce some graphic designations for various types of subcategories of $\mathcal{U}_{\geq}$ (see Table 1). These designations will be used, particularly, for the representation of the structure of the category of parabolic equations described below.

We shall use the term “meta-category” both for the category $\mathcal{U}_{\geq}$ and for its subcategories defined below to avoid confusion between $\mathcal{U}_{\geq}$ and “ordinary” categories which are objects of $\mathcal{U}_{\geq}$; and we shall use Gothic script for meta-categories except $\mathcal{U}_{\geq}$. One may view these meta-categories as a partial orders on the class of all subcategories of $\mathcal{U}$; we prefer category terminology here since this allows us to use category constructions for the interrelations between various partial orders.

Let us define wide subcategories $\mathfrak{W}$, $\mathfrak{F}$, $\mathfrak{F}_{\mathrm{I}}$, $\mathfrak{C}$, $\mathfrak{C}_{\mathrm{I}}$, $\mathfrak{D}$, and $\mathfrak{P}$ of meta-category Ug. Objects of them are categories, while arrows from $\mathcal{C}_1$ to $\mathcal{C}_2$ have a different meaning:

- in the meta-category $\mathfrak{W}$ it means that $\mathcal{C}_2$ is wide subcategory of $\mathcal{C}_1$,
- in the meta-category $\mathfrak{F}$ it means that $\mathcal{C}_2$ is full subcategory of $\mathcal{C}_1$,
- in the meta-category $\mathfrak{F}_{\mathrm{I}}$ it means that $\mathcal{C}_2$ is full under isomorphisms in $\mathcal{C}_1$,
- in the meta-category $\mathfrak{C}$ it means that $\mathcal{C}_2$ is closed subcategory of $\mathcal{C}_1$,
- in the meta-category $\mathfrak{C}_{\mathrm{I}}$ it means that $\mathcal{C}_2$ is closed under isomorphisms in $\mathcal{C}_1$,
- in the meta-category $\mathfrak{D}$ it means that $\mathcal{C}_2$ is dense subcategory of $\mathcal{C}_1$,

	$\mathfrak{W}$	Wide
	$\mathfrak{F}$	Full
	$\mathfrak{F}_{\mathrm{I}}$	Full under isomorphisms
	$\mathfrak{C}$	Close
	$\mathfrak{C}_{\mathrm{I}}$	Close under isomorphisms
	$\mathfrak{D}$	Dense
	$\mathfrak{P}$	Plentiful

Таблица 1. Basic meta-categories (arrows)

- in the meta-category $\mathfrak{P}$ it means that $\mathcal{C}_2$ is plentiful subcategory of $\mathcal{C}_1$,

We shall denote the intersections of these meta-categories by the concatenations of appropriate letters, for example: $\mathfrak{F}\mathfrak{D} = \mathfrak{F} \cap \mathfrak{D}$.

The following proposition is obvious.

Proposition 4. $\mathfrak{F}_{\mathrm{I}} \cap \mathfrak{P} = \mathfrak{F} \cap \mathfrak{P}$*;* $\mathfrak{C}_{\mathrm{I}} \cap \mathfrak{P} = \mathfrak{C}$*;* $\mathfrak{F} \cap \mathfrak{P} \cap \mathfrak{D} = \mathfrak{F} \cap \mathfrak{D}$.

Interrelations between "basic" meta-categories $\mathfrak{W}$, $\mathfrak{F}$, $\mathfrak{F}_{\mathrm{I}}$, $\mathfrak{C}$, $\mathfrak{C}_{\mathrm{I}}$, $\mathfrak{D}$, $\mathfrak{P}$ and their intersections ("composed" meta-categories) are represented on Fig. 5(a). Here an arrow means the predicate "to be subcategory of"; we shall call it the "meta-arrow". For example, meta-arrow from $\mathfrak{D}$ to $\mathfrak{W}$ means that $\mathfrak{W}$ is a subcategory of $\mathfrak{D}$. In the language of "ordinary" categories this meta-arrow means that the statement "$\mathcal{C}_2$ is wide in $\mathcal{C}_1$" implies that $\mathcal{C}_2$ is dense in $\mathcal{C}_1$. Everywhere on Fig. 5(a) a pair of meta-arrows with the same target means that this meta-category (target of these meta-arrows) is the intersection of two "top" meta-categories (sources of these meta-arrows). For example, $\mathfrak{F}\mathfrak{D} = \mathfrak{F}\mathfrak{P} \cap \mathfrak{P}\mathfrak{D}$.

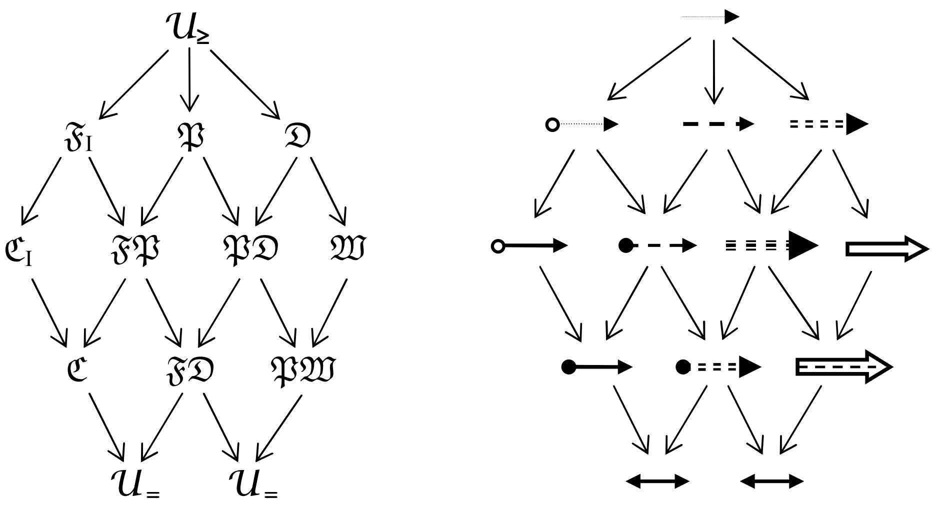
On Fig. 5(b) the same scheme is represented as on Fig. 5(a), but the letter names are replaced by the arrows of various types.

Instead of investigation of all or the simplest morphisms with the given source, we want to introduce a certain structure in $\mathcal{PDE}$, so that the position of an object in it gives an information about the morphisms from the object and about the kind of the simplest representatives of equivalence classes of the morphisms. In [8] we describe such a structure for the category of parabolic equations, choosing some subcategories of $\mathcal{PE}$ connected by the arrows from Fig. 5(b). Then we use this structure to describe the morphisms from nonlinear reaction-diffusion equation.

5. The category of parabolic equations

Let us consider the class $P\,(X, T, \Omega)$ of differential operators on a connected smooth manifold X, which depend additionally on a parameter t ("time"), locally having the form

$$Lu = \sum_{i,j} b^{ij}(t,x,u)u_{ij} + \sum_{i,j} c^{ij}(t,x,u)u_i u_j + \sum_i b^i(t,x,u)u_i + q(t,x,u),$$
$$x \in X,\ t \in T,\ u \in \Omega$$



a b

Рис. 5. Interrelations between basic meta-categories (arrows) and their intersections

in some neighborhood of each point, in some (and then arbitrary) local coordinates $\left(x^i\right)$ on X. Here subscript i denotes partial derivative with respect to x^i, quadratic form $b^{ij} = b^{ji}$ is positive definite, and $c^{ij} = c^{ji}$. Both T and Ω may be bounded, semibounded, or unbounded open intervals of $\mathbb{R}$.

D e f i n i t i o n 9. *The category $\mathcal{PE}$ of parabolic equations of the second order* is the full subcategory of $\mathcal{PDE}$, whose objects are pairs $\mathbf{A} = (N, E)$, $N = T \times X \times \Omega$ auch that X is a connected smooth manifold, T and Ω are open intervals, and E is an equation of the form $u_t = Lu$, $L \in P\,(X, T, \Omega)$ (more exactly, E is the extended version of the equation $u_t = Lu$, that is a closed submanifold of $J^2_{n+1}(T \times X \times \Omega)$, $n = \dim X$).

E x a m p l e 1. Let $\Phi_k(x)$, $x \in \mathbb{R}^3 - \{0\}$ be a spherical harmonic of the k-th order. Then the map $(t, x, u) \mapsto (t, |x|\,, u\,/\Phi_k(x)\,)$ defines the morphism in the category $\mathcal{PE}$ from the object $\mathbf{A}$ corresponding to equation $u_t = \Delta u$ and $X = \mathbb{R}^3 - \{0\}$, $T = \Omega = \mathbb{R}$, to the object $\mathbf{A}'$ corresponding to equation $u'_{t'} = u'_{x'x'} - k\,(k+1)\,x'^{-2}u'$ and $X' = \mathbb{R}_+$, $T' = \Omega' = \mathbb{R}$. One may assign to the set $\mathrm{Sol}(\mathbf{A}')$ of all solutions of the quotient equation the set $F^*(\mathrm{Sol}(\mathbf{A}'))$ of such solutions of the original equation that may be written in the form $u = \Phi_k(x)u'\,(t, |x|)$.

E x a m p l e 2. The following example shows that not every endomorphism in $\mathcal{PE}$ is an automorphism. Consider object $\mathbf{A}$, for which $X = S^1 = \mathbb{R} \bmod 1$, $T = \Omega = \mathbb{R}$, $E\colon u_t = u_{xx}$. Then the morphism from $\mathbf{A}$ to $\mathbf{A}$ defined by the map $(t, x, u) \mapsto (4t, 2x, u)$ has no inverse.

Theorem 1. *Every morphism in $\mathcal{PE}$ has the form*

$$(t, x, u) \mapsto \left(t'(t), x'\,(t, x)\,, u'(t, x, u)\right), \tag{5.2}$$

with submersive $t'(t)$, $x'(t, x)$, and $u'\,(t, x, u)$. Isomorphisms in $\mathcal{PE}$ are exactly diffeomorphisms of the form (5.2)*.*

P r o o f. Passing from the equation $u_t = Lu$ to the equation in the extended jet bundle for unknown submanifold $L \subset X \times T \times \Omega$ locally defined by the formula $f\,(t, x, u) = 0$, and expressing the derivatives of u by the corresponding derivatives of f, we obtain the following extended version of E:

$$f_t f_u^2 = \sum_{i,j} b^{ij}\left(f_{ij}f_u^2 - \left(f_{iu}f_j + f_{ju}f_i\right)f_u + f_i f_j f_{uu}\right) - \sum_{i,j} c^{ij} f_i f_j f_u + \sum_i b^i f_i f_u^2 - q f_u^3 \tag{5.3}$$

Suppose $F\colon \mathbf{A}\to\mathbf{A}'$ is a morphism in $\mathcal{PE}$, $N'=X'\times T'\times\Omega'$, and E' is defined by the equation

$$u'=\sum_{i',j'}B^{i'j'}\left(t',x',u'\right)u'_{i'j'}+\sum_{i',j'}C^{i'j'}\left(t',x',u'\right)u'_{i'}u'_{j'}+\sum_{i'}B^{i'}\left(t',x',u'\right)u'_{i'}+Q\left(t',x',u'\right).$$

Consider the extended analog of the last equation:

$$\begin{aligned}f'_{t'}f'^2_{u'}=\sum_{i',j'}B^{i'j'}\Big(f'_{i'j'}f'^2_{u'}-\left(f'_{i'u'}f'_{j'}+f'_{j'u'}f'_{i'}\right)f'_{u'}+f'_{i'}f'_{j'}f'_{u'u'}\Big)-\\-\sum_{i',j'}C^{i'j'}f'_{i'}f'_{j'}f'_{u'}+\sum_{i'}B^{i'}f'_{i'}f'^2_{u'}-Qf'^3_{u'},\end{aligned}\tag{5.4}$$

where $f'\left(t',x',u'\right)=0$ is the equation locally defining a submanifold L' of N'.

Recall that $F\colon\ (t,x,u)\mapsto\left(t',x',u'\right)$ is a morphism in $\mathcal{PE}$ if and only if for each point $\vartheta\in N$ and for each submanifold L' of N', $F(\vartheta)\in L'$, the following two conditions are equivalent:

- the 2-jet of L' at the point $F(\vartheta)$ satisfies (5.4)
- the 2-jet of $F^{-1}\left(L'\right)$ at the point ϑ satisfies (5.3).

In other words, the conditions "f' is solution of (5.4)" and "f is solution of (5.3)" should be equivalent when

$$f\left(t,x,u\right)=f'\left(t'\left(t,x,u\right),x'\left(t,x,u\right),u'\left(t,x,u\right)\right).$$

To find all such maps we use the following procedure:

1. Express derivatives of f in (5.3) through derivatives of f':

$$\frac{\partial f}{\partial t}=\frac{\partial f'}{\partial t'}\frac{\partial t'}{\partial t}+\frac{\partial f'}{\partial x'^{i'}}\frac{\partial x'^{i'}}{\partial t}+\frac{\partial f'}{\partial u'}\frac{\partial u'}{\partial t}$$

   and so on.
2. In the obtained identity substitute the combinations of the derivatives of f' for $\partial f'/\partial t'$ by formula (5.4). Then repeat this step for $\partial^2 f'/\partial t'^2$ in order to eliminate all derivatives with respect to t'. After reducing to common denominator, the transformed identity will have the form $\Phi=0$, where Φ is a rational function of partial derivatives of f' with respect to x' and u'. The coefficients $\phi_1,\dots,\phi_s$ of Φ are functions of 4-jet of the map F.
3. Solve the system $\phi_1=0,\dots,\phi_s=0$ of partial differential equations for a map F.

Let us realize this procedure. Note that we shall not write out function Φ completely. Instead we consider only some of its coefficients and then use the obtained information about F in order to simplify Φ step by step.

First note that the derivatives of the forth order arise only in term $\partial^2 f'/\partial t'^2$ when we fulfill the step 2 of the above procedure. Write this term before the final realization of step 2 for the sake of simplicity:

$$\begin{aligned}\Phi=\sum_{i,j}b^{ij}\left(t'_i t'_j f_u^2-t'_i t'_u f_j f_u-t'_j t'_u f_i f_u+t'^2_u f_i f_j\right)\frac{\partial^2 f'}{\partial t'^2}+\ldots=\\=\sum_{i,j}b^{ij}\left(t'_i f_u-t'_u f_i\right)\left(t'_j f_u-t'_u f_j\right)\frac{\partial^2 f'}{\partial t'^2}+\ldots\end{aligned}$$

The coefficient at $\partial^2 f'/\partial t'^2$ must be zero, and the quadratic form b^{ij} is positive definite. We get $t'_i f_u=t'_u f_i$, so

$$t'_i\left(f'_{t'}t'_u+f'_{x'}x'_u+f'_{u'}u'_u\right)=t'_u\left(f'_{t'}t'_i+f'_{x'}x'_i+f'_{u'}u'_i\right)$$

(here and below we use the notations

$$f'_{i'} = \frac{\partial f'}{\partial x'_{i'}}, \quad f'_{x'} x'_u = \sum_{j'} f'_{j'} x'^{j'}_u$$

and so on). Hence we obtain the following system of equations:

$$\begin{cases} t'_u u'_i = u'_u t'_i \\ t'_u x'_i = x'_u t'_i \end{cases} \tag{5.5}$$

One of the following three conditions holds:

1. $t'_u = 0$, $t'_x = 0$;
2. $t'_u = 0$, $t'_x \neq 0$;
3. $t'_u \neq 0$.

In the second case $u'_u = x'_u = 0$. Taking into account the equality $t'_u = 0$, we obtain a desired contradiction to the assumption that F is a submersion.

In the third case we get from (5.5) the identities $t'_x = \omega t'_u$, $u'_x = \omega u'_u$, $x'^{i'}_x = \omega x'^{i'}_u$, where $\omega = t'_x / t'_u$ is a section of $\pi^* T^* X$, $\pi\colon N = T \times X \times \Omega \to X$ is the natural projection, $\pi^* T^* X$ is the vector bundle over N induced by π from the cotangent bundle $T^* X$, and $\omega = \sum_i \omega_i(t, x, u) \mathrm{d}x^i$ in local coordinates. This implies that $f_x = \omega f_u$. Substituting the last formula to (5.3), we get

$$f_t = f_u \left[\sum_{i,j} b^{ij} \left(\frac{\partial \omega_i}{\partial x^j} - \omega_j \frac{\partial \omega_i}{\partial u} \right) - \sum_{i,j} c^{ij} \omega_i \omega_j + \sum_i b^i \omega_i - q \right].$$

Denote the expression in square brackets by $\zeta(t, x, u)$. Then $f_t = \zeta f_u$. Expressing derivatives of f in terms of derivatives of f', we obtain $t'_t = \zeta t'_u$, $x'_t = \zeta x'_u$, $u'_t = \zeta u'_u$. Consider the field of hyperplanes that kill the 1-form $\mathrm{d}t'$ in the tangent bundle TM (recall that $t'_u \neq 0$, so $\mathrm{d}t'$ is non-degenerated). The differential of F vanishes on these hyperplanes because $\mathrm{d}u' \wedge \mathrm{d}t' = \mathrm{d}x'^{i'} \wedge \mathrm{d}t' = 0$. Therefore $\operatorname{rank}(\mathrm{d}F) \leq 1$. Since $\dim N' \geq 3$, F can not be submersive, which contradicts the definition of an admitted map.

Finally, we see that only the first case is possible. Hence t' is a function of t, and $f'_{t'}$ may appear only in the representation of f_t. Let us look at the terms of Φ containing $(f'_{u'})^{-2}$:

$$\Phi = \sum_{i',j',k',l'} t'_t x'^{i'}_u x'^{j'}_u B'^{k'l'} f'_{i'} f'_{j'} f'_{k'} f'_{l'} f'_{u'u'} \left(f'_{u'}\right)^{-2} + \dots.$$

Substitution of any covector $\omega = \sum_{i'} \omega_{i'} \mathrm{d}x'^{i'} \in \Gamma(T^* X')$ to the expression

$$\sum_{i',j',k',l'} t'_t x'^{i'}_u x'^{j'}_u B'^{k'l'} \omega_{i'} \omega_{j'} \omega_{k'} \omega_{l'} = t'_t \left(\sum_{i'} x'^{i'}_u \omega_{i'} \right)^2 \left(\sum_{k',l'} B'^{k'l'} \omega_{k'} \omega_{l'} \right)$$

should give zero. The quadratic form $B'^{k'l'}$ is positive definite, so $\sum_{k',l'} B'^{k'l'} \omega_{k'} \omega_{l'} > 0$ when $\omega \neq 0$. Taking into account that F is submersive, we obtain $t'_t \neq 0$. Therefore $\sum_{i'} x'^{i'}_u \omega_{i'} = 0$ for any ω, that is $x'_u \equiv 0$. This implies $x' = x'(t, x)$, $t' = t'(t)$, which completes the proof. □

6. Comparison with the reduction by a symmetry group

As Remark 2 shows, our definition of morphism in $\mathcal{PDE}$ is a generalization of the reduction by a symmetry group. So we can obtain sets of solutions more general than the sets of group-invariant solutions provided by the group analysis of PDE (though our approach is more laborious owing to the non-linearity of the system of PDE describing a morphisms). Let us illustrate this by an example of a primitive morphism.

D e f i n i t i o n 10. A morphism $F\colon \mathbf{A} \to \mathbf{B}$ of a category $\mathcal{C}$ is called a *reducible in* $\mathcal{C}$ if there exist non-invertible morphisms $G\colon \mathbf{A} \to \mathbf{C}$, $H\colon \mathbf{C} \to \mathbf{B}$ in $\mathcal{C}$ such that $F = H \circ G$. Otherwise, a morphism is called *primitive in* $\mathcal{C}$.

Note that the reduction of PDE by a symmetry group defines a primitive morphism if and only if this group has no proper subgroups The reduction by any symmetry group that is not a discrete cyclic group of prime order may be always represented as a superposition of two nontrivial reductions, so the corresponding morphism is a superposition of two non-invertible morphisms and therefore is reducible. In particular, this situation takes place for any nontrivial connected Lie group.

However, the situation for morphisms is completely different. Even a morphism that decreases the number of independent variables by 2 or more may be primitive; below we present an example of such a morphism. In contrast, in the Lie group analysis we always have one-parameter subgroups of a symmetry group, so the morphism corresponding to a symmetry group is always reducible.

E x a m p l e 3. Consider the following morphism $F\colon \mathbf{A} \to \mathbf{B}$ in $\mathcal{PE}$:

- $\mathbf{A}$ is the heat equation $u_t = a(u)\Delta u$ posed on $X = \{(x, y, z, w) : z < w\} \subset \mathbb{R}^4$ equipped with the metric
$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & \alpha & \beta \\ 0 & \alpha & 1 & 0 \\ 0 & \beta & 0 & 1 \end{pmatrix},$$
where $\alpha = xe^w$, $\beta = xe^z$, $\gamma = 1 + \alpha^2 + \beta^2$, $a \notin \mathcal{A}_{\text{exp}} \cup \mathcal{A}_{\text{deg}}$. In the coordinate form $\mathbf{A}$ looks as
$$a^{-1}(u)u_t = u_{xx} + u_{yy} - 2\alpha u_{yz} - 2\beta u_{yw} + \left(1 + \alpha^2\right) u_{zz} +$$
$$+ 2\alpha\beta u_{zw} + \left(1 + \beta^2\right) u_{ww} + (\alpha\beta)_w\, u_z + (\alpha\beta)_z\, u_w.$$
- $\mathbf{B}$ is the heat equation $a^{-1}(u)u_t = u_{xx} + u_{yy}$ posed on $Y = \{(x, y)\} = \mathbb{R}^2$ equipped with Euclidean metric.
- The morphism F is defined by the map $(t, (x, y, z, w), u) \mapsto (t, (x, y), u)$.

This morphism decreases the number of independent variables by 2 and nevertheless is primitive in $\mathcal{PE}$.

Additional examples of morphisms that are not defined by any symmetry group of the given PDE, and also a detailed investigation of the case $\dim Y = \dim X - 1$, may be found in [6], [7].

REFERENCES

1. **Elkin V. I.** Reduction of nonlinear control systems. A differential geometric approach. Mathematics and its Applications, **472**, Kluwer Academic Publishers, Dordrecht, 1999.
2. **Krasil'shchik J., Vinogradov A. (eds.)** Symmetries and conservation laws for differential equations of mathematical physics. Translations of Mathematical Monographs, **182**, AMS, Providence, RI, 1999.
3. **Mac Lane S.** Categories for the working mathematician. Springer-Verlag, NY, 1998.
4. **Olver P. J.** Applications of Lie groups to differential equationsю Springer-Verlag, NY, 1993.
5. **Prokhorova M. F.** Modeling of solutions of heat equation and Stefan problem with dimension decrease // Russian Acad. Sci. Docl. Math. **58** (1), 1998. 88–90.
6. **Prokhorova M. F.** Modeling of heat equation and Stefan problem. Preprint VINITI N 347-V00, IMM UrBr Russian Acad. Sci., Ekaterinburg, 2000. 1–66 (in Russian).
7. **Prokhorova M. F.** Factorization of nonlinear heat equation posed on Riemann manifold // arXiv.org:math.AP/0108001, 2001. 1–9.
8. **Prokhorova M. F.** The structure of the category of parabolic equations. II // Trudy Instituta Matematiki i Mekhaniki UrO RAN, 2015. In print.

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