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FINITELY GENERATED LATTICES WITH COMPLETELY MODULAR ELEMENTS AMONG GENERATORS

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We look at the concept of a completely modular element of a lattice, which is the modular analog of the well-known concept of a neutral element of a lattice. It is proved that a lattice is modular if it is generated by three elements of which two are completely modular. Also it is shown that a lattice generated by n, $n > 3$, *completely modular elements must not necessarily be modular.*

Research on the structure of a lattice is not infrequently based on the idea of distinguishing elements of the lattice enjoying particular good properties such as being an atom, being a distributive, standard, or neutral element. Note that the definitions of a distributive element and of a standard element have been couched using the same scheme: in an equality expressing the distributive law, the universal quantifier is attached only to two elements out of three, while the remaining free element for which a proposition so constructed is true does acquire an appropriate name (see [1, p. 76]). If, however, in a distributive equality, the universal quantifier is attached to just one element then we obtain, for instance, the definition of a distributive pair of elements.

A similar approach applies to a modular quasi-identity. In particular, L. Wilcox in [3] came up with the following:

Definition 1. A pair (a, b) of elements of a lattice $\langle L; \wedge, \vee\rangle$ is said to be *modular* if

$$\forall x \in L : x \leq b \to (a \vee x) \wedge b = (a \wedge b) \vee x.$$

If, however, in the above formula, the universal quantifier is attached, not to one, but to two elements, then we face the concepts of a left-modular element and of a right-modular element.

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Definition 2. Let L be an arbitrary lattice.

(i) An element a is said to be *left modular* if

$$\forall x, b \in L : x \leq b \rightarrow (a \vee x) \wedge b = (a \wedge b) \vee x.$$

(ii) An element b is said to be *right modular* if

$$\forall a, x \in L : x \leq b \rightarrow (a \vee x) \wedge b = (a \wedge b) \vee x.$$

These elements can be defined differently (see [1, pp. 39, 75]); this, in particular, sheds light on the origins of the terms.*

(i) An element a is said to be *left modular* if a pair (a, b) is modular for any b.

(ii) An element b is said to be *right modular* if a pair (a, b) is modular for any a.

That the two definitions are equivalent is obvious.

It is not hard to see that the property of an element being left modular is self-dual, which, however, is untrue for an element with the property of being right modular.

Definition 3. An element of a lattice is said to be *coright modular* if it is right modular in the dual lattice.

The properties mentioned above are weaker versions of the following: distributivity of a lattice element, codistributivity, standardness, and costandardness. In fact, it is easy to verify that every costandard element is left modular. Since the property of being left modular is self-dual, a standard element of a lattice is also left modular. In turn, every codistributive element is right modular, while every distributive element is coright modular.

G. Birkhoff in [9] introduced the concept of a neutral element. We can say that a neutral element is one that possesses all the four properties at once—that is, distributivity, codistributivity, standardness, and costandardness. Actually, two properties, for example, standardness and codistributivity, are sufficient. (If these properties are projected onto modularity, then these are simultaneously the left and right modularities.) With modularity, however, not only are all the three properties pairwise distinct, but it is also the fact that even if an element has any two properties of the three then this does not necessarily imply that the element possesses the third. A simplest example in support is a well-known pentagon such as depicted in Fig. 1: the element a is right modular and coright modular but is not left modular; b is left modular and coright modular but is not right modular; c is left modular and right modular but is not coright modular. Naturally, therefore, we are interested in elements that possess all the three properties simultaneously.

Definition 4. An element of a lattice is said to be *completely modular* if it is simultaneously right modular, left modular, and coright modular.

*It is worth pointing out that here we have a substantial divergence in terminology. In a series of papers, for instance, in [4], right-modular elements are referred to as modular. In Russian literature, not infrequently, left-modular elements and right-modular elements are called modular elements and upper-modular elements, respectively [5, 6]. In studies dealing with subgroup and subring lattices, being modular refers to elements that are simultaneously coright modular and left modular [7, 8], etc. In the present paper, we adhere to the terminology adopted in [1].

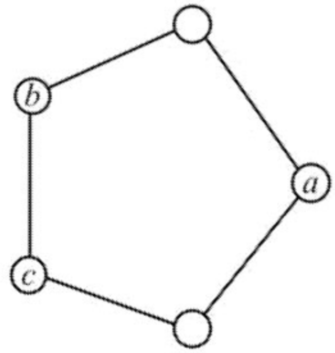

Fig. 1

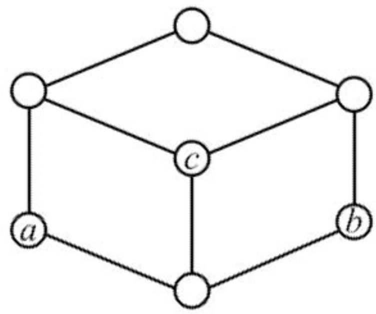

Fig. 2

The main property of a neutral element is the following: an element is neutral iff together with any other two elements of a lattice it generates a distributive sublattice.

Obtaining an exact analog of the above statement for a completely modular element is impossible: a nonmodular lattice such as in Fig. 2 is generated by elements a, b, and c, in which case c is completely modular. In addition, the elements a and b are right modular in this instance, for every atom in any lattice is automatically right modular. Hence a lattice generated by three right-modular elements does not need to be modular. If we pass to the dual of the above lattice, we see that a lattice generated by three coright modular elements, too, must not necessarily be modular. At the same time, a lattice generated by three left-modular elements is always modular [10].

Yet the following theorem is valid, which can be treated as a limited transfer of the properties of neutral elements to completely modular ones.

THEOREM 1. A lattice generated by three elements of which two are completely modular is modular.

The question arises whether a lattice generated by four or more completely modular elements is modular. An answer is given in

THEOREM 2. For any natural $n > 3$, there exists a nonmodular lattice L generated by n atoms each of which is completely modular in L.

The results above were announced in [11].

For brevity, we introduce some notation. Let $p(x, y, z)$ be a lattice polynomial in variables x, y, and z, i.e., an element of a free lattice generated by elements x, y, and z. Denote by $p^{\#}(x, y, z)$ a polynomial obtained from $p(x, y, z)$ by replacing the operation $\vee$ with $\wedge$, and conversely by replacing the operation $\wedge$ with $\vee$. We write $t(x, y, z)$ for a polynomial $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ and write $s(x, y, z)$ for a polynomial $t^{\#}(x, y, z)$. Clearly, $t(x, y, z)$ is the same element of a free lattice under any permutation of x, y, and z. The same is also true for $s(x, y, z)$. Let $f(x, y, z) = (x \vee (y \wedge z)) \wedge (y \vee z)$. A distributive inequality shows that $(x \vee (y \wedge z)) \wedge (y \vee z) \leq ((x \vee y) \wedge (x \vee z)) \wedge (y \vee z)$; i.e., $f(x, y, z) \leq t(x, y, z)$. In turn, a modular inequality yields $(x \vee (y \wedge z)) \wedge (y \vee z) \geq (x \wedge (y \vee z)) \vee (y \wedge z)$; i.e., $f(x, y, z) \geq f^{\#}(x, y, z)$. Thus $s(x, y, z) \leq f^{\#}(x, y, z) \leq f(x, y, z) \leq t(x, y, z)$ for any elements x, y, and z in any lattice.

Prior to proving Theorem 1, we give a number of statements, which are interesting in their own right.

LEMMA 1. Let x be some element of a lattice L.

(1) The element x is left modular if and only if $f(x, y, z) = f^{\#}(x, y, z)$ for any y and any z in L.

(2) If x is completely modular, then $f(y, z, x) = f^{\#}(y, z, x)$ for any y and any z in L.

(3) If $f(y, z, x) = f^{\#}(y, z, x)$ for any y and any z in L, then x is right modular and coright modular.

Proof. (1) The property of x being left modular and the relation $y \wedge z \leq y \vee z$ imply that $f(x, y, z) = (x \vee (y \wedge z)) \wedge (y \vee z) = (x \wedge (y \vee z)) \vee (y \wedge z) = f^{\#}(x, y, z)$. Conversely, if $y \leq z$, then $f(x, y, z) = (x \vee y) \wedge z = f^{\#}(x, y, z) = (x \wedge z) \vee y$. Now the left modularity of x follows from the definition.

(2) Let x be completely modular. Then $x \wedge f(y, z, x) = x \wedge (y \vee (z \wedge x)) \wedge (z \vee x) = x \wedge (y \vee (z \wedge x)) = (x \wedge y) \vee (z \wedge x) \leq s(x, y, z) \leq f^{\#}(y, z, x)$ in view of x being right modular. On the other hand, the coright modularity of x implies the inequality $x \vee f^{\#}(y, z, x) \geq f(y, z, x)$ which is dual to the previous. Now, by virtue of the left modularity of x and the inequality $f^{\#}(y, z, x) \leq f(y, z, x)$, we obtain $f(y, z, x) = f(y, z, x) \wedge (x \vee f^{\#}(y, z, x)) = (f(y, z, x) \wedge x) \vee f^{\#}(y, z, x) = f^{\#}(y, z, x)$.

(3) For $x \leq z$, we have $f(y, z, x) = (y \vee x) \wedge z = f^{\#}(y, z, x) = (y \wedge z) \vee x$. This means that x is coright modular. That x is right modular follows from duality considerations.

Note that the statements reverse to those in items (2) and (3) of Lemma 1 are invalid. For the statement in (2), a proper example is the above-mentioned pentagon: the element a specified in the pentagon is not left modular, but it is not hard to verify that $f(x, y, a) = f^{\#}(x, y, a)$ for any of the lattice elements x and y. For the statement in (3), an example of a lattice refuting the reverse statement is given in Fig. 3. It is easy to verify that the element x in that lattice is right modular and coright modular; yet the elements $f(y, z, x)$ (denoted f for brevity) and $f^{\#}(y, z, x)$ (which coincides with z) are different.

LEMMA 2. An element x of a lattice L is simultaneously left modular and coright modular

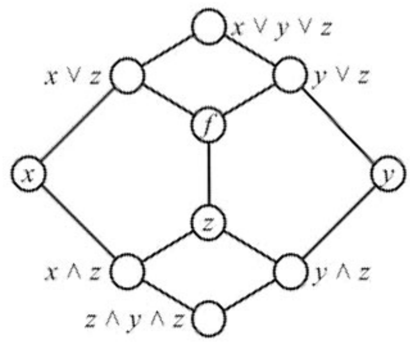

Fig. 3

if and only if $t(x, y, z) = (x \wedge (y \vee z)) \vee (y \wedge (x \vee z))$ for any elements y and z in L.

Proof. Since $y \vee z \geq y \wedge (x \vee z)$ and $x \vee z \geq x$, we have $(x \wedge (y \vee z)) \vee (y \wedge (x \vee z)) = (x \vee (y \wedge (x \vee z)) \wedge (y \vee z) = ((x \vee y) \wedge (x \vee z)) \wedge (y \vee z) = t(x, y, z)$. Conversely, if $x \leq z$, then $t(x, y, z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z) = (x \vee y) \wedge z = (x \wedge (y \vee z)) \vee (y \wedge (x \vee z)) = x \vee (y \wedge z)$. That x is coright modular follows from the definition. If, however, $y \leq z$, then $t(x, y, z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z) = (x \vee y) \wedge z = (x \wedge (y \vee z)) \vee (y \wedge (x \vee z)) = (x \wedge z) \vee y$; i.e., the element x is left modular.

LEMMA 3. Let y and z be arbitrary elements of a lattice.

(1) If an element x is coright modular, then $(x \vee y) \wedge (z \vee x) = x \vee t(x, y, z)$.

(2) If an element x is left modular and coright modular, then $(x \vee y) \wedge (y \vee z) = y \vee t(x, y, z)$.

Proof. (1) Since $x \leq (x \vee y) \wedge (z \vee x)$, the coright modularity of x shows that $x \vee t(x, y, z) = x \vee ((x \vee y) \wedge (z \vee x) \wedge (y \vee z)) = ((x \vee y) \wedge (z \vee x)) \wedge (x \vee y \vee z) = (x \vee y) \wedge (z \vee x)$.

(2) Lemma 2 implies that $y \vee t(x, y, z) = y \vee (x \wedge (y \vee z)) \vee (y \wedge (x \vee z)) = y \vee (x \wedge (y \vee z))$. The left modularity of x and the inequality $y \leq y \vee z$ yield $y \vee (x \wedge (y \vee z)) = (x \vee y) \wedge (y \vee z)$.

Proof of Theorem 1. Let L be a lattice generated by an arbitrary element c and by completely modular elements a and b. In [12], eleven relations were pointed out whose satisfaction for generating elements of a lattice implies being modular for the lattice. We list these relations using the notation introduced above:

$$f(a, b, c) = f^{\#}(a, b, c), \tag{1}$$

$$f(b, c, a) = f^{\#}(b, c, a), \tag{2}$$

$$f(c, a, b) = f^{\#}(c, a, b), \tag{3}$$

$$\begin{gathered} t(a, b, c) = ((a \wedge (b \vee c)) \vee ((a \vee b) \wedge c)) \\ \wedge((b \wedge (a \vee c)) \vee ((b \vee a) \wedge c)) \wedge ((a \wedge (c \vee b)) \vee ((a \vee c) \wedge b)), \end{gathered} \tag{4}$$

$$\begin{gathered} s(a, b, c) = ((a \vee (b \wedge c)) \wedge ((a \wedge b) \vee c)) \\ \vee((b \vee (a \wedge c)) \wedge ((b \wedge a) \vee c)) \vee ((a \vee (c \wedge b)) \wedge ((a \wedge c) \vee b)), \end{gathered} \tag{5}$$

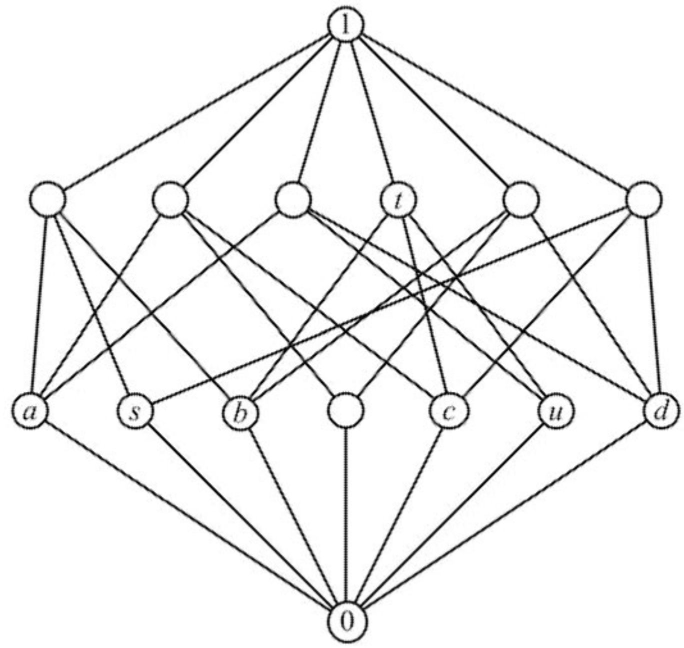

Fig. 4

$$(a \vee b) \wedge (a \vee c) = a \vee t(a, b, c), \tag{6}$$

$$(b \vee a) \wedge (b \vee c) = b \vee t(a, b, c), \tag{7}$$

$$(c \vee a) \wedge (c \vee b) = c \vee t(a, b, c), \tag{8}$$

$$(a \wedge b) \vee (a \wedge c) = a \wedge s(a, b, c), \tag{9}$$

$$(b \wedge a) \vee (b \wedge c) = b \wedge s(a, b, c), \tag{10}$$

$$(c \wedge a) \vee (c \wedge b) = c \wedge s(a, b, c). \tag{11}$$

Our present goal is to verify that the eleven relations follow from the property of being completely modular for a and b. The first three relations derive from Lemma 1; relations (4) and (5) are consequences of Lemma 2 and its dual; the last six relations follow from Lemma 3 and dual statements.

Proof of Theorem 2. First we give an example that demonstrates the validity of the theorem for $n = 4$. A diagram of a corresponding lattice is presented in Fig. 4. In the lattice, generating elements are a, b, c, and d, and these generators are all completely modular. Yet the lattice itself is not modular—elements $t \vee s$, s, t, u, and $t \wedge s$ form a pentagon.

LEMMA 4. Let L be a lattice generated by n completely modular atoms. Then a direct product of L and a two-element chain is a lattice generated by $n + 1$ completely modular atoms.

Proof. Denote by $a_1, a_2, \ldots, a_n$ the n completely modular atoms generating the lattice L, in which 0 is a least element and 1 is a greatest element. It is easy to verify that

$(a_1, 0), (a_2, 0), \ldots, (a_n, 0)$, and $(0, 1)$ are generating elements of the above direct product; moreover, these are completely modular atoms of the product.

The proof of Theorem 2 for $n > 4$ is completed by applying Lemma 4 as many times as necessary.

Possibly, there exists a nonmodular lattice generated by four completely modular elements that contains less elements than the lattice depicted in Fig. 4. Still our example is interesting in virtue of the fact that the lattice in question is atomistic* and its generating completely modular elements are atoms. If, however, in an atomistic lattice of finite length, all atoms are completely modular (and even only coright modular), then such a lattice is modular [4].

REFERENCES

1. M. Stern, *Semimodular Lattices. Theory and Applications, Encyclopedia Math. Appl.*, **73**, Cambridge Univ. Press, Cambridge (1999).
2. S. Maeda, "On distributive pairs in lattices," *Acta Math. Hung.*, **45**, Nos. 1/2, 133-140 (1985).
3. L. R. Wilcox, "Modularity in the theory of lattices," *Ann. Math. (2)*, **40**, 490-505 (1939).
4. S. Maeda, "On finite-modular atomistic lattices," *Alg. Univers.*, **12**, 76-80 (1981).
5. B. M. Vernikov and M. V. Volkov, "Modular elements of the lattice of semigroup varieties. II," *Contrib. Gen. Alg.*, **17**, G. Dorfer et al. (eds.), Verlag Johannes Heyn, Klagenfurt (2006), pp. 173-190.
6. B. M. Vernikov, "Upper-modular elements of the lattice of semigroup varieties. II," *Fundam. Prikl. Mat.*, **14**, No. 7, 43-51 (2008).
7. R. Schmidt, "Modulare Untergruppen endlicher Gruppen," *Ill. J. Math.*, **13**, 358-377 (1969).
8. R. K. Amayo and J. Schwarz, "Modularity in Lie algebras," *Hiroshima Math. J.*, **10**, 311-322 (1980).
9. G. Birkhoff, "Neutral elements in general lattices," *Bull. Am. Math. Soc.*, **46**, 702-705 (1940).
10. A. G. Gein and M. P. Shushpanov, "Sublattice generated by modular elements," *Algebra and Linear Optimization, Proc. Int. Conf.*, Yekaterinburg (2012), p. 47.
11. A. G. Gein and M. P. Shushpanov, "Lattices generated by completely modular elements," *Mal'tsev Readings* (2012), p. 52.
12. A. G. Gein and M. P. Shushpanov, "Defining relations of a free modular lattice of rank 3," *Izv. Vysch. Uch. Zav., Ser. Mat.*, No. 10, 69-72 (2013).

*In Russian literature, the term a 'point lattice' is also used to refer to this concept.