FILTERS BASED ON AGGREGATION OPERATORS.
Part 2. The Kolmogorov filters[1]

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Abstract — In this work, a new class of nonlinear filters called the generalized Kolmogorov filters for image processing is introduced. These filters are based on Kolmogorov-Maslov-Matkowski aggregation operators. We show that a large body of nonlinear filters proposed to date constitute a proper subset of the generalized Kolmogorov filters.

ФИЛЬТРЫ,ОСНОВАННЫЕ НА АГРЕГАЦИОННЫХ ОПЕРАТОРАХ
Часть 2. Фильтры Колмогорова

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Аннотация — В данной работе вводится новый класс нелинейных фильтров для обработки мультиспектральных изображений, названный векторными медианными агрегационными фильтрами. Они основываются на произвольной паре агрегационных операторов и одном селекционном правиле. Мы показываем, что большое множество нелинейных фильтров, предложенных к настоящему времени являются собственным подмножеством множества новых фильтров.

I. Introduction

According to Kolmogorov [1] a sequence of functions $\mathbf{Aggeg}_N$ defines a regular type of average if the following conditions are satisfied:

1) $\mathbf{Agg}_N(x_1, x_2, ..., x_N)$ is continuous and monotone in each variable; to be definite, we assume that $\mathbf{Agg}_N$ is increasing in each variable.
2) $\mathbf{Agg}_N(x_1, x_2, ..., x_N)$ is a symmetric function.
3) The average of identical numbers is equal to their common value: $\mathbf{Agg}_N(x, x, ..., x) = x$.
4) A group of values can be replaced by their own average, without changing the overall average:
$$\mathbf{Agg}_{N+M}(x_1, x_2, ..., x_N, y_1, y_2, ..., y_M) =$$
$$= \mathbf{Agg}_{N+M}(m, m, ..., m, y_1, y_2, ..., y_M),$$

where $m = \mathbf{Agg}_N(x_1, x_2, ..., x_N)$.

Proposition 1. (Kolmogorov). If conditions 1)–4) are satisfied, the average $\mathbf{Agg}_N(x_1, x_2, ..., x_N)$ is of the form:

$$\mathbf{Kolm}(K \mid x_1, x_2, ..., x_N) = K^{-1}\left[\sum_{i=1}^{N} \frac{1}{N} K(x_i)\right],$$

where K is a strictly monotone continuous function in the extended real line.

We list below a few particular cases of means:

1) Arithmetic mean ($K(x) = x$):

$$\mathbf{Mean}(x_1, x_2, ..., x_N) = \frac{1}{N}\sum_{i=1}^{N} x_i. \qquad (1)$$

2) Geometric mean ($K(x) = \log(x)$):

$$\mathbf{Geo}(x_1, x_2, ..., x_N) = \sqrt[N]{\prod_{i=1}^{N} x_i} = \exp\left(\frac{1}{n}\sum_{i=1}^{N} \ln x_i\right). \qquad (2)$$

3) Harmonic mean ($K(x) = x^{-1}$):

$$\mathbf{Harm}(x_1, x_2, ..., x_N) = \frac{1}{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{x_i}} = \frac{N}{\sum_{i=1}^{N}\frac{1}{x_i}}. \qquad (3)$$

4) A very notable particular case corresponds to the function $K(x) = x^p$. We obtain then a quasi arithmetic (power or Hölder) mean of the form:

$$\mathbf{Power}_p(x_1, x_2, ..., x_N) = \left(\frac{1}{N}\sum_{i=1}^{N} x_i^p\right)^{\frac{1}{p}}. \qquad (4)$$

This family is particularly interesting, because it generalizes a group of common means, only by changing the value of p .

II. Main Part

Let us introduce the observation model and notion used throughout the paper. We consider noise images of the

$$f(\mathbf{x}) = s(\mathbf{x}) + \mu(\mathbf{x}), \qquad (5)$$

Where $s(\mathbf{x})$ is the original grey-level image and $\eta(\mathbf{x})$ denotes the noise introduced into $s(\mathbf{x})$ to produce the corrupted image $f(\mathbf{x})$ and where $\mathbf{x} = (i, j) \in \mathbf{Z}^2$ (or $\mathbf{x} = (i, j, k) \in \mathbf{Z}^3$) is a 2D (or 3D) coordinates that belong to the image domain and represent the pixel location. If $\mathbf{x} \in \mathbf{Z}^2, \mathbf{Z}^3$, then $f(\mathbf{x}), s(\mathbf{x}),$ and $\eta(\mathbf{x})$ are 2D and 3D images, respectively.

The aim of image enhancement is to reduce the noise as much as possible or to find a method which, given $s(\mathbf{x})$, derives an image $\hat{s}(\mathbf{x})$ as close as possible

1

to the original $s(\mathbf{x})$, subject to a suitable optimality criterion. In a 2D standard mean filter with a square window $\left[\mathrm{M}_{(i,j)}(m,n)\right]_{m=-r,n=-r}^{m=+r,n=+r}$ of size $N=(2r+1)\times(2r+1)$ is located at (i,j) the mean replace the central pixel $\hat{s}(i,j)=\underset{(m,n)\in M_{(i,j)}}{\mathbf{Mean}}\left\{f(m,n)\right\}$, where $\hat{s}(i,j)$ is the filtered image, $\left\{f(m,n)\right\}_{(m,n)\in M_{(i,j)}}$ is image block of the fixed size N extracted from f by moving window $\mathrm{M}_{(i,j)}$ at the position (i,j). When this filter is modified as arithmetic mean of K-deformed pixels we have the Kolmogorov filter:

$$\hat{s}(i,j)=\underset{(m,n)\in M_{(i,j)}}{\mathbf{Kolm}}\left\{K\mid f(m,n)\right\}=$$

$$=K^{-1}\left[\sum_{(m,n)\in M_{(i,j)}}^{N}\frac{1}{N}K\left(f(m,n)\right)\right]. \quad (6)$$

Maslov [2] modified the fourth axiom as follows: for any set of weights w_i, any set of numbers x_i, $i=1,2,...,N$, and any number a, the following equality holds:

$$\mathbf{Maslov}\left(K\mid x_1+a,x_2+a,...,x_N+a\right)=$$

$$=K^{-1}\left[\sum_{i=1}^{N}\bar{w}_iK\left(x_i+a\right)\right]=Ca+K^{-1}\left[\sum_{i=1}^{N}\bar{w}_iK\left(x_i\right)\right] \quad (7)$$

where C is a number not depending on a. Then, the following assertion is valid [2].

Proposition 2. Condition (7) implies that the function $K(x)$ has the form

$$K(x)=A\mathbf{exp}(-\beta x),\quad \text{or}\quad K(x)=Ax+D,$$

where the numbers β and D are nonzero.

According to Proposition 2, the nonliner averaging of a random variables $x_1,x_2,...,x_N$ corresponding to a set of weights w_i has the form

$$\mathbf{Maslov}^{\beta}_{\bar{w}_1,\bar{w}_2,...,\bar{w}_N,}\left(x_1,x_2,...,x_N\right)=\frac{1}{\beta}\ln\left[\sum_{i=1}^{N}\bar{w}_ie^{-\beta x_i}\right].$$

Aggregation filters associated with Maslov means form a new β-parameterized family of nonlinear filters

$$\hat{s}(i,j)=\underset{(m,n)\in M_{(i,j)}}{\mathbf{Maslov}^{\beta}}\left\{f(m,n)\right\}=$$

$$=\frac{1}{\beta}\ln\left[\frac{1}{N}\sum_{(m,n)\in M_{(i,j)}}^{N}\bar{w}(m,n)e^{-\beta\cdot f(m,n)}\right]. \quad (8)$$

It is known [3] that under some natural assumptions on real functions $K_1,K_2,...,K_M$ the function $\mathbf{Agg}\left(K_1,K_2,...,K_N\mid x_1,x_2,...,x_N\right)$ defined by

$$\mathbf{Agg}\left(K_1,K_2,...,K_N\mid x_1,x_2,...,x_N\right)=$$

$$=\left(K_1+K_2+...+K_N\right)^{-1}\left[\sum_{i=1}^{N}\bar{w}_iK_i\left(x_i\right)\right]$$

is a generalization of the Kolmogorov mean. We will call it the Matkowski mean and denote as

$$\mathbf{Matkow}_{\bar{w}_1,\bar{w}_2,...,\bar{w}_N}\left(K_1,K_2,...,K_N\mid x_1,x_2,...,x_N\right)=$$

$$=\left(K_1+K_2+...+K_N\right)^{-1}\left[\sum_{i=1}^{N}\bar{w}_iK_i\left(x_i\right)\right].$$

Aggregation filters associated with the Matkowski means form a new family of nonlinear filters

$$\hat{s}(i,j)=\underset{(m,n)\in M_{(i,j)}}{\mathbf{Matkow}}\left\{K_{m,n}\mid f(m,n)\right\}=$$

$$=\left(\sum_{(m,n)\in M_{(i,j)}}^{N}K_{m,n}\right)^{-1}\left[\sum_{(m,n)\in M_{(i,j)}}^{N}\bar{w}(m,n)K_{m,n}\left(f(m,n)\right)\right].$$

We define new generalized Kolmogorov, Maslov and Matkowski means as follows:

$$\mathbf{GenKolm}\left(K\mid x_1,x_2,...,x_N\right)=K^{-1}\left[\mathbf{Aggreg}\left(K\left(x_i\right)\right)\right],$$

$$\mathbf{GenMaslov}^{\beta}\left(x_1,x_2,...,x_N\right)=\frac{1}{\beta}\ln\left[\mathbf{Aggreg}\left(e^{-\beta x_i}\right)\right],$$

$$\mathbf{GenMatkow}_{\bar{w}_1,...,\bar{w}_N}\left(K_1,...,K_N\mid x_1,x_2,...,x_N\right)=$$

$$=\left(K_1+...+K_N\right)^{-1}\left[\mathbf{Aggreg}\left(K_i\left(x_i\right)\right)\right].$$

Aggregation filter associated with these means are aggregations of K - deformed pixels:

$$\hat{s}(i,j)=\underset{(m,n)\in M_{(i,j)}}{\mathbf{GenKolm}}\left\{K\mid f(m,n)\right\}=$$

$$=K^{-1}\left[\underset{(m,n)\in M_{(i,j)}}{\mathbf{Agg}}\left(K\left(f(m,n)\right)\right)\right],$$

$$\hat{s}(i,j)=\underset{(m,n)\in M_{(i,j)}}{\mathbf{GenMaslov}^{\beta}}\left\{f(m,n)\right\}=\frac{1}{\beta}\ln\left[\mathbf{Agg}\left(e^{-\beta\cdot f(m,n)}\right)\right],$$

$$\hat{s}(i,j)=\underset{(m,n)\in M_{(i,j)}}{\mathbf{GenMatkow}}\left\{K_{m,n}\mid f(m,n)\right\}=$$

$$=\left(\sum_{(m,n)\in M_{(i,j)}}^{N}K_{m,n}\right)^{-1}\left[\mathbf{Agg}\left(K_{m,n}\left(f(m,n)\right)\right)\right].$$

We show that a large body of nonlinear filters proposed to date constitute a proper subset of the generalized Kolmogorov filters.

III. Conclusion

We developed a new theoretical framework for image filtering using Kolmogorov aggregation operators. The main goal of the work is to show that Kolmogorov aggregation operators can be used to solve problems of image filtering in a natural and effective manner.

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IV. References

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