FILTERS BASED ON AGGREGATION OPERATORS.
Part 3. The Heron filters

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Abstract — In this work, a new class of nonlinear filters called the generalized arithmetic and median Heronian filters for image processing is introduced. These filters are based on Heronian mean.

ФИЛЬТРЫ,ОСНОВАННЫЕ НА АГРЕГАЦИОННЫХ ОПЕРАТОРАХ
Часть 3. Фильтры Герона

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Аннотация — В данной работе вводится новый класс нелинейных фильтров для обработки мультиспектральных изображений, названный обобщенными арифметическими и медианными фильтрами Герона. Эти фильтры основываются на среднем по Герону.

I. Введение

The classical Heronian mean and median definition of two positive real numbers a and b are

$$\mathbf{MeanHeron}(a,b)=\left(a+\sqrt{ab}+b\right)/3=$$
$$\left(\sqrt{aa}+\sqrt{ab}+\sqrt{bb}\right)/3, \qquad (1)$$
$$\mathbf{MedHeron}(a,b)=\mathbf{Med}\left\{\sqrt{aa},\sqrt{ab},\sqrt{bb}\right\}.$$

(*Hero of Alexandria* is the Greek mathematician)

Let $(x_1,x_2,...,x_N)$ be an N-tuple of positive real numbers. An obvious way to generalize Eq.(1) is by including inside the parentheses the square roots of all possible products of two elements of *a* (with repetition) and dividing the whole by the total number of such products, $N(N+1)/2$.

Definition 1. One can write this as:

$$\mathbf{MeanHeron}_2(x_1,x_2,...,x_N)=\frac{2}{N(N+1)}\sum_{i}\le\sum_{j}\sqrt{x_i x_j},$$

$$\mathbf{MedHeron}_2(x_1,x_2,...,x_N)=\mathbf{Med}\left[\left\{\sqrt{x_i x_j}\right\}_{i\le j}\right].$$

What we want to do here is more radical and consists of summing up the k-*th* roots of all possible distinct products of k elements of $(x_1,x_2,...,x_N)$, again with repetition. The number of all such products corresponds to extracting k elements from a bag of N, with replacement, where $C_{N+k-1}^k=\frac{(N+k-1)!}{k!(N-1)!}$ is the binomial coefficient. This determines the normalization factor and leads to the definition:

$$\mathbf{MeanHeron}_k(x_1,x_2,...,x_N)=$$
$$=\frac{2}{C_{N+k-1}^k}\sum_{i_1}\le\sum_{i_2}\le...\le\sum_{i_2}\sqrt[k]{x_{i_1}x_{i_2}\cdots x_{i_k}}. \qquad (2)$$

Obviously,

$$\mathbf{MedHeron}_k(x_1,x_2,...,x_N)=$$
$$=\mathbf{Med}\left[\left\{\sqrt[k]{x_{i_1}x_{i_2}\cdots x_{i_k}}\right\}_{i_1\le i_2\le...\le i_k}\right]. \qquad (3)$$

II. Main Part

Let us introduce the observation model and notion used throughout the paper. We consider noise images of the

$$f(\mathbf{x})=s(\mathbf{x})+\mu(\mathbf{x}),$$

where $s(\mathbf{x})$ is the original grey-level mage and $\eta(\mathbf{x})$ denotes the noise introduced into $s(\mathbf{x})$ to produce the corrupted image $f(\mathbf{x})$ and where $\mathbf{x}=(i,j)\in\mathbf{Z}^2$ (or $\mathbf{x}=(i,j,k)\in\mathbf{Z}^3$) is a 2D (or 3D) coordinates that belong to the image domain and represent the pixel location. If $\mathbf{x}\in\mathbf{Z}^2,\mathbf{Z}^3$ then $f(\mathbf{x}),s(\mathbf{x}),\ \eta(\mathbf{x})$ are 2D or 3D images, respectively.

The aim of image enhancement is to reduce the noise as much as possible or to find a method which, given $s(\mathbf{x})$, derives an image $\hat{s}(\mathbf{x})$ as close as possible to the original $s(\mathbf{x})$, subject to a suitable optimality criterion.

In a 2D standard linear and median filters with a square window $\left[\mathrm{M}_{(i,j)}(m,n)\right]_{m=-r,n=-r}^{m=+r,n=+r}$ of size

$N=(2r+1)\times(2r+1)$ is located at (i,j) the mean and median replace the central pixel

$$\hat{s}(i,j)=\underset{(m,n)\in M_{(i,j)}}{\mathbf{Mean}}\left\{f(m,n)\right\},$$

$$\hat{s}(i,j)=\underset{(m,n)\in M_{(i,j)}}{\mathbf{Med}}\left\{f(m,n)\right\},$$

where $\hat{s}(i,j)$ is the filtered image, $\left\{f(m,n)\right\}_{(m,n)\in M_{(i,j)}}$ is image block of the fixed size $N=(2r+1)\times(2r+1)$ extracted from f by moving window $\mathrm{M}_{(i,j)}$ at the position (i,j), **Mean** and **Med** are the mean (average) and median operators. When those filters are modified as follows

$$\hat{s}(i,j)=\underset{(k,l)\in M(i,j)}{\mathbf{Aggreg}}\left\{f(k,l)\right\},$$

it becomes an aggregation digital filter, where **Aggreg** is an aggregation operator.

Let us introduce $(k+1)$-valued function on mask $\sigma:M\to\{0,1,...,k\}$. Obviously, $w(\sigma)=\sum_{(n,m)\in M}\sigma(n,m)$ is the weight of a function σ and $w(\sigma)\in\{0,1,...,Nk\}$, where $N=|M|$. If $B_{k+1}^{N}=\left\{\sigma\mid\sigma:M\to\{0,1,2,...,k\}\right\}$ is the set of all $(k+1)$-valued functions, then

$$B_{k+1}^{N}={}^{0}B_{k+1}^{N}\cup...\cup{}^{r}B_{k+1}^{N}\cup...\cup{}^{Nk}B_{k+1}^{N}=\bigcup_{r=0}^{Nk}{}^{r}B_{k+1}^{N},$$

where

$${}^{i}B_{k+1}^{N}=\left\{{}^{i}\sigma\mid\left({}^{i}\sigma:M\to\{0,1,...,k\}\right)\&\left(w({}^{i}\sigma)=i\right)\right\}$$

is the set of all $(k+1)$-valued function with weight r and $\left|{}^{r}B_{k+1}^{N}\right|$ - cardinality of the set ${}^{r}B_{k+1}^{N}$. Now we define a product of pixels

$$f^{\,{}^{r}\sigma}:=\prod_{(n,m)\in M}\left(f(n,m)\right)^{{}^{r}\sigma(n,m)}$$

associated with $(k+1)$-valued function ${}^{r}\sigma$. Using this product we define generalized aggregation Heronian filter as

$$\hat{s}(i,j)=\underset{(m,n)\in M_{(i,j)}}{\mathbf{GenHeron}^{r}}\left\{f(m,n)\right\}=$$

$$=\underset{\text{over all } {}^{r}\sigma\in{}^{r}B_{2}^{N}}{\mathbf{Aggreg}}\left(\sqrt[r]{\prod_{(n,m)\in M_{(i,j)}}\left(f(n,m)\right)^{{}^{r}\sigma(n,m)}}\right).$$

In particular cases we have

1) the arithmetic r-Heronian filter

$$\hat{s}(i,j)=\underset{(m,n)\in M_{(i,j)}}{\mathbf{MeanHeron}^{r}}\left\{f(m,n)\right\}=$$

$$\underset{\text{over all } {}^{r}\sigma\in{}^{r}B_{2}^{N}}{\mathbf{Mean}}\left(\sqrt[r]{\prod_{(n,m)\in M_{(i,j)}}\left(f(n,m)\right)^{{}^{r}\sigma(n,m)}}\right)=$$

$$=\frac{1}{\left|{}^{r}B_{k+1}^{N}\right|}\sum_{{}^{r}\sigma\in{}^{r}B_{2}^{N}}\sqrt[r]{\prod_{(n,m)\in M_{(i,j)}}\left(f(n,m)\right)^{{}^{r}\sigma(n,m)}},$$

2) the median r-Heronian filter

$$\hat{s}(i,j)=\underset{(m,n)\in M_{(i,j)}}{\mathbf{MedHeron}^{r}}\left\{f(m,n)\right\}=$$

$$=\underset{(n,m)\in M_{(i,j)}}{\mathbf{Med}}\left[\left\{\sqrt[r]{\left(f(n,m)\right)^{{}^{r}\sigma(n,m)}}\right\}_{\text{over all } {}^{r}\sigma\in{}^{r}B_{k+1}^{N}}\right].$$

3) the Kolmogorov Heronian filter

$$\hat{s}(i,j)=\underset{(m,n)\in M_{(i,j)}}{\mathbf{KolmHeron}^{r}}\left\{K\mid f(m,n)\right\}=$$

$$=K^{-1}\left[\underset{\text{over all } {}^{r}\sigma\in{}^{r}B_{k+1}^{N}}{\mathbf{Mean}}\left(K\left(\sqrt[r]{\prod_{(n,m)\in M_{(i,j)}}\left(f(n,m)\right)^{{}^{r}\sigma(n,m)}}\right)\right)\right]=$$

$$=K^{-1}\left[\frac{1}{\left|{}^{r}B_{k+1}^{N}\right|}\sum_{{}^{r}\sigma\in{}^{r}B_{k+1}^{N}}K\left(\sqrt[r]{\prod_{(n,m)\in M_{(i,j)}}\left(f(n,m)\right)^{{}^{r}\sigma(n,m)}}\right)\right],$$

where $r\le k$.

a) b)

c) $\underset{(m,n)\in M_{(i,j)}}{\mathbf{MedHeron}^{2}}$ d) $\underset{(m,n)\in M_{(i,j)}}{\mathbf{MeanHeron}^{2}}$

Fig. 1. Original (a) and noise (b) images. Denoised images (c) and (d).
Рис. 1. Исходное (a) и зашумленное (b) изображения. Отфильтрованные изображения (c) и (d).

In Fig.1 we present examples of $\underset{(m,n)\in M_{(i,j)}}{\mathbf{MeanHeron}^{2}}$ - and $\underset{(m,n)\in M_{(i,j)}}{\mathbf{MedHeron}^{2}}$ -filtering for $N=|M|=5\times5$.

III. Conclusion

We developed a new theoretical framework for image filtering using the Heronian aggregation operators. The main goal of the work is to show that Heronian aggregation operators can be used to solve problems of image filtering in a natural and effective manner.

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IV. References

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