EPIGROUP VARIETIES OF FINITE DEGREE

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Аннотация. An epigroup variety is said to be a variety of finite degree [a variety of degree n] if all its nilsemigroups are nilpotent [of degree $\leq n$ and n is the least number with such a property]. We characterize epigroup varieties of finite degree and of an arbitrary given degree n in a language of identities and in terms of minimal forbidden subvarieties.

1. Introduction and summary

A semigroup S is called an *epigroup* if some power of each element of S lies in a subgroup of S. The class of epigroups is quite wide. It includes, in particular, all *completely regular* semigroups (i. e. unions of groups) and all *periodic* semigroups (i. e. semigroups in which every element has an idempotent power). Epigroups are intensively studied in the literature under different names since the end of 1950's. An overview of results obtained here is given in the fundamental work by L. N. Shevrin [8] and its survey [9].

It is natural to consider epigroups as *unary semigroups*, i. e. semigroups equipped with an additional unary operation. This operation is defined by the following way. If S is an epigroup and $a \in S$ then some power of a lies in a maximal subgroup of S. We denote this subgroup by G_a. The unit element of G_a is denoted by a^ω. It is well known (see [8], for instance) that the element a^ω is well defined and $aa^\omega = a^\omega a \in G_a$. We denote the element inverse to aa^ω in G_a by $\overline{a}$. The map $a \mapsto \overline{a}$ is the unary operation on S mentioned above. The element $\overline{a}$ is called *pseudoinverse* to a. Throughout this article we consider epigroups as algebras with two operations, namely multiplication and pseudoinversion. In particular, this allows us to say about varieties of epigroups as algebras with these two operations. An investigation of epigroups in the framework of the theory of varieties was promoted by L. N. Shevrin in the mentioned article [8]. An overview of first results obtained here may be found in [11, Section 2].

An examination of semigroup varieties shows that properties of a variety are depended in an essential degree by properties of nilsemigroups belonging to the variety. More precisely, we have in mind the question, whether a variety contains non-nilpotent nilsemigroups; if it is not the case then what about nilpotency degrees of nilsemigroups in a variety? This gives natural the following definitions. A semigroup variety $\mathcal{V}$ is called a variety of *finite degree* if all nilsemigroups in

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$\mathcal{V}$ are nilpotent. If $\mathcal{V}$ has a finite degree then it is said to be a *variety of degree* n if nilpotency degrees of all nilsemigroups in $\mathcal{V}$ does not exceed n and n is the least number with such a property. Semigroup varieties of finite degree and some natural subclasses of this class of varieties were investigated in [5, 7, 13, 14] and other articles (see also Section 8 in the survey [10]).

It is well known and may be easily verified that, in a periodic semigroup, pseudoinversion may be written by using of multiplication only. Indeed, if an epigroup satisfies the identity

$$x^p = x^{p+q} \tag{1.1}$$

for some natural numbers p and q then the identity

$$\overline{x} = x^{(p+1)q-1} \tag{1.2}$$

holds in this epigroup. If $p > 1$ then the simpler formula

$$\overline{x} = x^{pq-1} \tag{1.3}$$

is valid. This means that periodic varieties of epigroups may be identified with periodic varieties of semigroups. Semigroup varieties of finite degree are periodic, whence they may be considered as epigroup varieties. It seems to be natural to expand the notions of varieties of finite degree or of degree n to all epigroup varieties. Definitions of epigroup varieties of finite degree or degree n repeat literally definitions of the same notions for semigroup varieties.

In [7, Theorem 2], semigroup varieties of finite degree were characterized in several ways. In particular, it was proved there that a semigroup variety $\mathcal{V}$ has a finite degree if and only if it satisfies an identity of the form

$$x_1 \cdots x_n = w \tag{1.4}$$

for some natural n and some word w of length $> n$. Moreover, the proof of this result easily implies that $\mathcal{V}$ has a degree $\le n$ if and only if it satisfies an identity of the form (1.4) for some word w of length $> n$. For varieties of degree 2, this equational characterization was essentially specified in [3, Lemma 3]. Namely, it was verified there that a semigroup variety has degree ≤ 2 if and only if it satisfies an identity of the form $xy = w$ where w is one of the words $x^{m+1}y$, xy^{m+1} or $(xy)^{m+1}$ for some natural m. In [14, Proposition 2.11], analogous specification of the mentioned above result of [7] was obtained for semigroup varieties of degree $\le n$ with arbitrary n (see Proposition 2.1 and Corollary 2.2 below). The objective of this article is to expand the mentioned results of [7, 14] on epigroup varieties.

In order to formulate our results, we need some definitions and notation. We denote by F the free unary semigroup. The unary operation on F will be denoted by $\overline{}$. Elements of F are called *unary words* or simply *words*. A *semigroup word* is a word that does not contain the unary operation. If $w \in F$ then $\ell(w)$ stands for the length of w; here we assume that the length of any non-semigroup word is infinite. As usual, a pair of identities $wx = xw = w$ where the letter x does not occur in the word w is written as the symbolic identity $w = 0$. Note that this notation is justified because a semigroup with such identities has a zero element and all values of the word w in this semigroup are equal to zero. Further, let Σ be a system of identities written in the language of unary semigroups (that is, the language that consists of one associative binary operation and one unary operation). Then K_Σ stands for the class of all epigroups satisfying Σ (here we treat the unary operation from our language as the pseudoinversion). The class K_Σ is not obliged to be a variety because it maybe not closed under taking of infinite Cartesian products (see [9, Subsection 2.3] or Example 2.15 below, for instance). Note that this class is a generalized variety in the sense of [1]. A complete classification of identity system Σ such that K_Σ is a variety is provided by Proposition 2.16 below. If Σ has this property then we will write $\mathbf{V}[\Sigma]$ alongwith (and in the same sense as) K_Σ. It is evident that if the class K_Σ consists of periodic epigroups (in particular, of nilsemigroups) then it is a periodic semigroup variety, and therefore is an epigroup variety. Thus, the notation $\mathbf{V}[\Sigma]$ is correct in this case. We often use this observation below without any additional references. Put

$$\mathcal{F} = \mathbf{V}[x^2 = 0,\, xy = yx],$$
$$\mathcal{F}_k = \mathbf{V}[x^2 = x_1 \cdots x_k = 0,\, xy = yx]$$

where k is an arbitrary natural number. The main result of the article is the following

Theorem 1.1. *For an epigroup variety $\mathcal{V}$, the following are equivalent:*

1) *$\mathcal{V}$ is a variety of finite degree;*
2) *$\mathcal{V} \not\supseteq \mathcal{F}$;*
3) *$\mathcal{V}$ satisfies an identity of the form* (1.4) *for some natural n and some unary word w with $\ell(w) > n$;*
4) *$\mathcal{V}$ satisfies an identity of the form*

$$x_1 \cdots x_n = x_1 \cdots x_{i-1} \cdot \overline{\overline{x_i \cdots x_j}} \cdot x_{j+1} \cdots x_n \tag{1.5}$$

for some i, j and n with $1 \le i \le j \le n$.

As we will seen below, the proof of this theorem easily implies the following

Corollary 1.2. *Let n be an arbitrary natural number. For an epigroup variety $\mathcal{V}$, the following are equivalent:*

1) *$\mathcal{V}$ is a variety of degree $\le n$;*
2) *$\mathcal{V} \not\supseteq \mathcal{F}_{n+1}$;*
3) *$\mathcal{V}$ satisfies an identity of the form* (1.4) *for some unary word w with $\ell(w) > n$;*
4) *$\mathcal{V}$ satisfies an identity of the form* (1.5) *for some i and j with $1 \le i \le j \le n$.*

It is well known that an epigroup variety has degree 1 if and only if it satisfies the identity

$$x = \overline{\overline{x}} \tag{1.6}$$

(see Lemma 2.6 below). Besides that, it is evident that a variety has degree 1 if and only if it does not contain the variety of semigroups with zero multiplication, i. e. the variety $\mathcal{F}_2$. The equivalence of the claims 1), 2) and 4) of Corollary 1.2 generalizes these known facts.

The article consists of three sections. Section 2 contains definitions, notation and auxiliary results we need, while Section 3 is devoted to the proof of Theorem 1.1 and Corollary 1.2.

2. PRELIMINARIES

First of all, we formulate results about semigroup varieties of finite degree obtained in the articles [7, 14].

Proposition 2.1. *For a semigroup variety $\mathcal{V}$, the following are equivalent:*

- a) *$\mathcal{V}$ is a variety of finite degree;*
- b) *$\mathcal{V} \not\supseteq \mathcal{F}$;*
- c) *$\mathcal{V}$ satisfies an identity of the form* (1.4) *for some natural n and some semigroup word w with $\ell(w) > n$;*
- d) *$\mathcal{V}$ satisfies an identity of the form*

$$x_1 \cdots x_n = x_1 \cdots x_{i-1} \cdot (x_i \cdots x_j)^{m+1} \cdot x_{j+1} \cdots x_n \tag{2.1}$$

for some m, n, i and j with $1 \le i \le j \le n$. □

The equivalence of the claims a)–c) of this statement was proved in [7, Theorem 2], while the equivalence of the claims a) and d) immediately follows from [14, Proposition 2.11].

Corollary 2.2. *Let n be a natural number. For a semigroup variety $\mathcal{V}$, the following are equivalent:*

- a) *$\mathcal{V}$ is a variety of degree $\le n$;*
- b) *$\mathcal{V} \not\supseteq \mathcal{F}_{n+1}$;*
- c) *$\mathcal{V}$ satisfies an identity of the form* (1.4) *for some semigroup word w with $\ell(w) > n$;*
- d) *$\mathcal{V}$ satisfies an identity of the form* (2.1) *for some m, i and j with $1 \le i \le j \le n$.* □

The equivalence of the claims a)–c) easily follows from the proof of [7, Theorem 2], while the equivalence of the claims a) and d) is verified in [14, Proposition 2.11].

Now we formulate several simple and well known facts (see [8, 9], for instance).

Lemma 2.3. *If S is an epigroup and $x \in S$ then the equalities*

$$x\,\overline{x} = (x\,\overline{x})^2 = \overline{x\,\overline{x}}, \tag{2.2}$$

$$x\,\overline{x} = \overline{x}\,x = x^{\omega}, \tag{2.3}$$

$$x^{\omega}x = xx^{\omega} = \overline{\overline{x}}, \tag{2.4}$$

$$\overline{x} = \overline{x^2}\,x = x\,\overline{x^2}, \tag{2.5}$$

$$\overline{x^n} = \overline{x}^{\,n}, \tag{2.6}$$

$$\overline{\overline{\overline{x}}} = \overline{x} \tag{2.7}$$

hold where n is an arbitrary natural number. □

The equalities (2.3) show that it is correct to use the expression v^{ω} in epigroup identities as a short form of the term $v\,\overline{v}$. So, the equalities (2.2)–(2.7) are identities valid in arbitrary epigroup. We need the following generalization of the identities (2.3).

Corollary 2.4. *An arbitrary epigroup satisfies the identities*

$$x^n\,\overline{x}^{\,n} = \overline{x}^{\,n}x^n = x^\omega \tag{2.8}$$

for any natural number n.

Доказательство. Let S be an epigroup and $x \in S$. The identities (2.3) and the fact that x^ω is an idempotent in S imply that

$$x^n\,\overline{x}^{\,n} = \overline{x}^{\,n}x^n = (x\,\overline{x}\,)^n = (x^\omega)^n = x^\omega.$$

Corollary is proved. □

Lemma 2.5. *Every nil-epigroup satisfies the identity* $\overline{x} = 0$. □

Recall that a semigroup is called *completely regular* if it is a union of groups. Evidently, every completely regular semigroup is an epigroup. The operation of pseudoinversion on a completely regular semigroup coincides with the operation of taking of the element inverse to a given element x in the maximal subgroup that contains x. The latter operation is the intensively examined unary operation on the class of completely regular semigroups (see the book [6] or Section 6 of the survey [11], for instance). Thus, varieties of completely regular semigroups are epigroup varieties.

Lemma 2.6. *For an epigroup variety* $\mathcal{V}$*, the following are equivalent:*

a) $\mathcal{V}$ *is completely regular;*
b) $\mathcal{V}$ *is a variety of degree* 1*;*
c) $\mathcal{V}$ *satisfies the identity* (1.6). □

We denote by $\mathrm{Gr}\,S$ the set of all group elements of the epigroup S.

Lemma 2.7. *If* S *is an epigroup,* $x \in S$ *and* $x^n \in \mathrm{Gr}\,S$ *for some natural* n *then* $x^m \in \mathrm{Gr}\,S$ *for every* $m \geq n$. □

We need some additional definitions and notation. By F^1 we denote the unary semigroup F with the unit element (the empty word) adjoined. If $w \in F$ then $c(w)$ denotes the set of all letters occurring in the word w, while $t(w)$ stands for the last letter of w. The symbol $\equiv$ denotes the equality relation on the unary semigroups F and F^1. An identity $u = v$ is called a *semigroup identity* [a *mixed* identity, a *strictly unary* identity] if both the words u and v [exactly one of these words, none of these words] are semigroup words.

The following simple observation is formulated for convenience of references.

Lemma 2.8. *If an epigroup variety* $\mathcal{V}$ *satisfies a semigroup identity of the form* $u = v$ *with* $\ell(u) \neq \ell(v)$ *then* $\mathcal{V}$ *is periodic.*

Доказательство. It is well known that a semigroup variety is periodic if and only if it satisfies an identity of the form (1.1). Suppose that a variety satisfies a semigroup identity of the form $u = v$ with $\ell(u) \neq \ell(v)$. Let x be a letter. Substituting x to all letters from $c(u) \cup c(v)$ in the identity $u = v$, we obtain an identity of the form (1.1). □

A semigroup word w is called *linear* if any letter occurs in w at most one time. Recall that an identity of the form

$$x_1x_2\cdots x_n = x_{1\pi}x_{2\pi}\cdots x_{n\pi}$$

where π is a non-trivial permutation on the set $\{1,2,\dots,n\}$ is called *permutational.*

Lemma 2.9. *If an epigroup variety $\mathcal{V}$ satisfies a non-trivial identity of the form* (1.4) *then either this identity is permutational or $\mathcal{V}$ is a variety of degree $\le n$.*

Доказательство. If the word w contains the operation of pseudonversion then every nilsemigroup in $\mathcal{V}$ satisfies the identity $x_1\cdots x_n = 0$ by Lemma 2.5. Therefore, $\mathcal{V}$ is a variety of degree $\le n$ in this case. If w is a semigroup word and $\ell(w) > n$ then $\mathcal{V}$ is periodic by Lemma 2.8. Then it may be considered as a variety of semigroups. According to Corollary 2.2, this means that $\mathcal{V}$ is a variety of degree $\le n$. Suppose now that $\ell(w) \le n$. If $c(w) \ne \{x_1,\dots,x_n\}$ then $x_i \notin c(w)$ for some $1 \le i \le n$. One can substitute x_i^2 to x_i in (1.4). Then we obtain the identity $x_1\cdots x_{i-1}x_i^2x_{i+1}\cdots x_n = w$. Put $w' \equiv x_1\cdots x_{i-1}x_i^2x_{i+1}\cdots x_n$. Then $x_1\cdots x_n = w = w'$ holds in $\mathcal{V}$. Thus, $\mathcal{V}$ satisfies the identity $x_1\cdots x_n = w'$ and $\ell(w') > n$. As above, we may apply now Lemma 2.8 and Corollary 2.2 with the conclusion that $\mathcal{V}$ has degree $\le n$. Finally, if $c(w) = \{x_1,\dots,x_n\}$ then the fact that $\ell(w) \le n$ implies that $\ell(w) = n$, whence the word w is linear. Therefore, the identity (1.4) is permutational in this case. ☐

Put $\mathcal{P} = \mathbf{V}[xy = x^2y,\ x^2y^2 = y^2x^2]$. We need the following

Lemma 2.10. *If the variety $\mathcal{P}$ satisfies a non-trivial identity of the form* (1.4) *then $n > 1$ and $w \equiv w'x_n$ for some word w' with $c(w') = \{x_1,\dots,x_{n-1}\}$.*

Доказательство. It is well known and easy to check that the variety $\mathcal{P}$ is generated by the semigroup

$$P = \langle a, e \mid e^2 = e,\ ea = a,\ ae = 0\rangle = \{e, a, 0\}.$$

This semigroup is finite, whence it is an epigroup. Note that $\overline{e} = e$ and $\overline{a} = 0$. Suppose that $c(w) \ne \{x_1,\dots,x_n\}$. Then there is a letter x that occurs in one part of the indentity (1.4) but does not occur in the other one. Substituting 0 to x and e to all other letters occurring in the identity, we obtain the wrong equality $e = 0$. Therefore, $c(w) = \{x_1,\dots,x_n\}$. Substitute now a to x_n and e to all other letters occurring in the identity (1.4). The left part of the equlity we obtain equals a. We denote the right part of this equality by b. Thus, P satisfies the equality $a = b$. If the unary operation applies to the letter x_n in the word w or $t(w) \ne x_n$ then $b = 0$. But $a \ne 0$ in P. Therefore, $w \equiv w'x_n$. If $x_n \in c(w')$ then $b = 0$ again, thus $c(w') = \{x_1,\dots,x_{n-1}\}$. Finally, the word w' is non-empty because the identity (1.4) is non-trivial. Therefore, $n > 1$. ☐

Put $\mathcal{C} = \mathbf{V}[x^2 = x^3,\ xy = yx]$. The unary semigroup variety generated by an epigroup S is denoted by var S. Clearly, if the semigroup S is finite then var S is a variety of epigroups. The following statement was formulated without proof in [15, Theorem 3.2][1]. We provide the proof here for the sake of completeness.

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[1] There is some inaccuracy in the formulation of this assertion in [15]: it contains the words 'left ideal' rather than 'right ideal'.

Proposition 2.11. *Let $\mathcal{V}$ be an epigroup variety. For an arbitrary epigroup $S \in \mathcal{V}$, the set* $\mathrm{Gr}\ S$ *is a right ideal in S if and only if the variety $\mathcal{V}$ does not contain the varieties $\mathcal{C}$ and $\mathcal{P}$.*

Доказательство. Necessity. It is well known and easy to check that the varietiy $\mathcal{C}$ is generated by the semigroup

$$C = \langle a, e \mid e^2 = e,\ ae = ea = a,\ a^2 = 0\rangle = \{e, a, 0\}.$$

This semigroups is finite, whence it is an epigroups. Let S be one of the epigroups C and P. Then $\mathrm{Gr}\ S = \{e, 0\}$ and $ea = a \notin \mathrm{Gr}\ S$. We see that $\mathrm{Gr}\ S$ is not a right ideal in S, whence $C, P \notin \mathcal{V}$. Therefore, $\mathcal{C}, \mathcal{P} \nsubseteq \mathcal{V}$.

Sufficiency. Let S be an epigroup in $\mathcal{V}$ such that $\mathrm{Gr}\ S$ is not a right ideal in S. Then there are elements $x \in \mathrm{Gr}\ S$ and $y \in S$ with $xy \notin \mathrm{Gr}\ S$. Put $e = x^\omega$ and $a = xy$. Since $x \in \mathrm{Gr}\ S$, we have $ex = x$, and therefore $ea = exy = xy = a$. Let A be the subepigroup in S generated by the elements e and a. The equality $ea = a$ implies that every element in A equals to either e or a^k or $a^m e$ for some natural numbers k and m. Let now J be the ideal in A generated by the element ae. Clearly, any element in J equals to either a^k with $k > 1$ or $a^m e$. If $a \notin J$ then the Rees quotient epigroup A/J is isomorphic to the epigroup P. But this is impossible because $\mathrm{var}\ P = \mathcal{P} \nsubseteq \mathcal{V}$. Therefore, $a \in J$, whence either $a = a^k$ for some $k > 1$ or $a = a^m e$ for some natural m. In the former case we have $a \in \mathrm{Gr}\ S$, contradicting the choice of the elements x and y. It remains to consider the latter case. Then $ae = (a^m e)e = a^m e^2 = a^m e = a$. Let K be the ideal in A generated by the element a^2. It is easy to see that every element in K equals to a^k for some $k > 1$. It is clear that $a \notin K$ because $a \in \mathrm{Gr}\ S$ otherwise. Then the equalities $ea = a$ and $ae = a$ show that the Rees quotient epigroup A/K is isomorphic to the epigroup C. But this is not the case because $\mathrm{var}\ C = \mathcal{C} \nsubseteq \mathcal{V}$. □

Note that semigroup varieties with the property that, for any its member S, the set $\mathrm{Gr}\ S$ is an ideal or right ideal of S were examined in the article [12].

An epigroup variety $\mathcal{V}$ is called a *variety of epigroups with completely regular nth power* if, for any $S \in \mathcal{V}$, the epigroup S^n is completely regular. Put $\overleftarrow{\mathcal{P}} = \mathbf{V}[xy = xy^2,\ x^2y^2 = y^2x^2]$.

Lemma 2.12. *An epigroup variety of degree $\le n$ is a variety of epigroups with completely regular nth power if and only if it does not contain the varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$.*

Доказательство. Necessity. Let $\mathcal{V}$ be a variety of epigroups with completely regular nth power. In view of Lemma 2.6 $\mathcal{V}$ satisfies the identity

$$x_1 \cdots x_n = \overline{\overline{x_1 \cdots x_n}}. \tag{2.9}$$

But Lemma 2.10 and the dual statement imply that this identity is false in the varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$.

Sufficiency. Let $\mathcal{V}$ be a variety of epigroups of degree $\le n$ that does not contain the varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$. Further, let $S \in \mathcal{V}$ and $J = \mathrm{Gr}\ S$. Clearly, the variety $\mathcal{C}$ is not a variety of finite degree, whence $\mathcal{V} \nsupseteq \mathcal{C}$. Thus $\mathcal{V}$ contains none of the varieties $\mathcal{C}$, $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$. Now we may apply Proposition 2.11 and the dual statement with the conclusion that J is an ideal in S. If $x \in S$ then $x^n \in J$ for some n. This means that the Rees quotient semigroup S/J is a nilsemigroup. Since $\mathcal{V}$ is a variety of degree $\le n$, this means that the epigroup S/J satisfies the identity $x_1x_2\cdots x_n = 0$. In other words, if $x_1, x_2, \dots, x_n \in S$ then $x_1x_2\cdots x_n \in J$. Therefore, $S^n \subseteq J$, whence the epigroup S^n is completely regular. □

It is well known (see [8, 9], for instance) that the class of all epigroups is not a variety. In other words, the variety of unary semigroups generated by this class contains not only epigroups. Denote this variety by $\mathcal{EPI}$. We note that an identity basis of the variety $\mathcal{EPI}$ is known. This result was announced in 2000 by Zhil'tsov [16], and its proof was rediscovered recently by Mikhailova [4] (some related results can be found in [2]).

The number of occurrences of multiplication or unary operation in a word w is called the *weight* of w.

Lemma 2.13. *Let w be a non-semigroup word depending on a letter x only. Then the variety $\mathcal{EPI}$ satisfies an identity*

$$w = x^p\,\overline{x}^{\,q} \tag{2.10}$$

for some $p \ge 0$ and some positive integer q.

Доказательство. We use induction on the weight of w.

Induction base. If weigth of w equals 1 then $w \equiv \overline{x}$ and the requirement conclusion is evident.

Induction step. Suppose that the weight of the word w is $i > 1$. Further considerations are divided into two cases.

Case 1: $w \equiv w_1w_2$ where the weight of the words w_1 and w_2 is lesser than i. Obviously, at least one of the words w_1 or w_2 contains the unary operation. It suffices to consider the case when the word w_1 is non-semigroup. By the induction assumption, the identity $w_1 = x^s\,\overline{x}^{\,t}$ holds in $\mathcal{EPI}$ for some $s \ge 0$ and some positive integer t. If the word w_2 contains the unary operation then, by the induction assumption, the identity $w_2 = x^m\,\overline{x}^{\,k}$ holds in $\mathcal{EPI}$ for some $m \ge 0$ and some $k > 0$. If, otherwise, the word w_2 is a semigroup one then $w_2 \equiv x^r$ for some r. In any case, we may apply the identity (2.3) and conclude that the class $\mathcal{EPI}$ satisfies the identity (2.10).

Case 2: $w \equiv \overline{w_1}$ where the weight of the word w_1 is lesser than i. If the word w_1 is a semigroup one then $w_1 \equiv x^r$ for some r. Taking into account the identity (2.6), we have that the variety $\mathcal{EPI}$ satisfies the identity $w \equiv \overline{x^r} = \overline{x}^{\,r}$ here. If, otherwise, the word w_1 contains the unary operation then, by the induction assumption, the identity $w_1 = x^s\,\overline{x}^{\,t}$ holds in $\mathcal{EPI}$ for some $s \ge 0$ and some $t > 0$. If $s > t$ then

$$\begin{aligned}
w \equiv \overline{w_1} &= \overline{x^s\,\overline{x}^{\,t}} && \\
&= \overline{x^{s-t}x^t\,\overline{x}^{\,t}} && \\
&= \overline{x^{s-t}(x\,\overline{x}\,)^t} && \text{by (2.3)}\\
&= \overline{x^{s-t}(x\,\overline{x}\,)^{s-t}} && \text{by (2.2)}\\
&= \overline{x^{s-t}(x^{\omega})^{s-t}} && \text{by (2.3)}\\
&= \overline{(xx^{\omega})^{s-t}} && \text{by (2.4)}\\
&= \overline{\left(\overline{\overline{x}}\,\right)^{s-t}} && \text{by (2.4)}\\
&= \left(\overline{\overline{\overline{x}}}\,\right)^{s-t} && \text{by (2.6)}\\
&= \overline{x}^{\,s-t} && \text{by (2.7).}
\end{aligned}$$

If $s = t$ then

$$\begin{aligned}
w \equiv \overline{w_1} &= \overline{x^s\,\overline{x}^{\,s}} && \\
&= \overline{(x\,\overline{x}\,)^s} && \text{by (2.3)}\\
&= \overline{x\,\overline{x}} && \text{by (2.2)}\\
&= x\,\overline{x} && \text{by (2.2).}
\end{aligned}$$

Finally, if $s < t$ then

$$\begin{aligned}
w \equiv \overline{w_1} &= \overline{x^s\,\overline{x}^{\,t}} && \\
&= \overline{x^s\,\overline{x}^{\,s}\,\overline{x}^{\,t-s}} && \\
&= \overline{(x\,\overline{x}\,)^s\,\overline{x}^{\,t-s}} && \text{by (2.3)}\\
&= \overline{(x\,\overline{x}\,)^{t-s}\,\overline{x}^{\,t-s}} && \text{by (2.2)}\\
&= \overline{(x\,\overline{x}^{\,2})^{t-s}} && \text{by (2.2)}\\
&= \overline{\left(x\,\overline{x^2}\,\right)^{t-s}} && \text{by (2.6)}\\
&= \overline{\overline{x}^{\,t-s}} && \text{by (2.5)}\\
&= \left(\overline{\overline{x}}\,\right)^{t-s} && \text{by (2.6)}\\
&= (xx^{\omega})^{t-s} && \text{by (2.4)}\\
&= (x^2\,\overline{x}\,)^{t-s} && \text{by (2.3)}\\
&= x^{2(t-s)}\,\overline{x}^{\,t-s} && \text{by (2.3).}
\end{aligned}$$

So, we have proved that the variety $\mathcal{EPI}$ satisfies an identity of the form (2.10) in any case. □

As usual, we say that an epigroup S has an *index* n if $x^n \in \mathrm{Gr}\ S$ for any $x \in S$ and n is the least number with such a property. Following [8, 9], we denote the class of all epigroups of index $\le n$ by $\mathcal{E}_n$. It is well known that $\mathcal{E}_n$ is an epigroup variety (see [8, Proposition 6] or [9, Proposition 2.10], for instance).

Corollary 2.14. *If a class of unary semigroups K is contained in $\mathcal{EPI}$ and satisfies a mixed identity then K consists of epigroups and $K \subseteq \mathcal{E}_n$ for some n.*

Доказательство. Suppose that K satisfies a mixed identity $u = v$. Substitute some letter x to all letters occurring in this identity. Then we obtain an identity of the form $x^n = w$ for some positive integer n and some non-semigroup word w depending on the letter x only. According to Lemma 2.13, the variety $\mathcal{EPI}$ satisfies an identity of the form (2.10). Therefore, the class K satisfies the identities

$$\begin{aligned} x^n &= w \\ &= x^p\,\overline{x}^{\,q} && \text{by (2.10)} \\ &= (x^p\,\overline{x}^{\,q-1})\,\overline{x}^{\,2}x && \text{by (2.5)} \\ &= (x^p\,\overline{x}^{\,q})x\,\overline{x} && \text{by (2.3)} \\ &= x^n x\,\overline{x} && \text{by (2.10)} \\ &= x^{n+1}\,\overline{x}. \end{aligned}$$

So, the identity $x^n = x^{n+1}\,\overline{x}$ holds in the class K. It is well known (see [9, p. 334], for instance) that if a unary semigroup $S \in K$ satisfies this identity then S is an epigroup of index $\le n$. ☐

Let Σ be a system of identities written in the language of unary semigroups. As we have already noted, the class K_Σ is not obliged to be a variety. This claim is confirmed by the following

Example 2.15. Put $N_k = \langle a \mid a^{k+1} = 0\rangle = \{a, \dots, a^k, 0\}$ for any natural k. The semigroup N_k is finite, therefore it is an epigroup. Put

$$N = \prod_{k\in\mathbb{N}} N_k.$$

Obviously, the semigroup N is not an epigroup because, for example, no power of the element $(a, \dots, a, \dots)$ belongs to a subgroup. Note that the epigroup N_k is commutative for any k. We see that the class K_Σ with $\Sigma = \{xy = yx\}$ is not a variety.

If w is a semigroup word then $\ell_x(w)$ denote the number of occurrences of the letter x in this word. Recall that a semigroup identity $u = v$ is called *balanced* if $\ell_x(u) = \ell_x(v)$ for any letter x. The following statement gives a complete description of identity systems Σ such that K_Σ is a variety.

Proposition 2.16. *Let Σ be a system of identities written in the language of unary semigroups. The following are equivalent:*

1) *K_Σ is a variety;*
2) *Σ implies in the class of all epigroups some mixed identity;*
3) *Σ contains either a semigroup non-balanced identity or a mixed identity.*

Доказательство. 1) $\longrightarrow$ 3) Suppose that each identity in Σ is either balanced or strictly unary. We note that the epigroup N_k from Example 2.15 satisfies any balanced identity and any strictly unary one. In particular, any identity from Σ

holds in the epigroup N_k. Hence $N_k \in K_\Sigma$ for any k. Example 2.15 shows that the class K_Σ is not a variety.

3) $\longrightarrow$ 2) The case when Σ contains a mixed identity is evident. Suppose now that Σ contains a semigroup non-balanced identity $u = v$. Then $\ell_x(u) \ne \ell_x(v)$ for some letter x. If $\ell(u) = \ell(v)$ then we substitute x^2 to x in $u = v$. As a result, we obtain a semigroup non-balanced identity $u' = v'$ such that K satisfies $u' = v'$, $\ell_x(u') \ne \ell_x(v')$ and $\ell(u') \ne \ell(v')$. This allows us to suppose that $\ell(u) \ne \ell(v)$. Substitute some letter x to all letters occurring in this identity. We obtain an identity of the form (1.1). As it was mentioned above, this identity implies in the class of all epigroups the identity (1.2). It remains to note that this identity is mixed.

2) $\longrightarrow$ 1) Obviously, the class K_Σ is closed under taking of subepigroups and homomorphisms. It remains to prove that it is closed under taking of Cartesian products. Let $\{S_i \mid i \in I\}$ be an arbitrary set of epigroups from K_Σ. Consider the semigroup

$$S = \prod_{i\in I} S_i.$$

According to Corollary 2.14, there exists a number n such that $x^n \in \mathrm{Gr}\ S$ for any $S \in K_\Sigma$ and any $x \in S$. In particular, the epigroup S_i for any $i \in I$ has this property. But then the semigroup S also satisfies this condition, i. e. S is an epigroup. Obviously, any identity from Σ holds in the epigroup S. Therefore, $S \in K_\Sigma$ and we are done. $\square$

Let Σ be a system of identities written in the language of unary semigroups. We denote a variety of unary semigroups that satisfy identity system Σ by $\mathrm{var}\ \Sigma$. Denote the set of all identities that hold in any epigroup by Δ. Thus $\mathcal{EPI} = \mathrm{var}\ \Delta$. Let $\mathrm{var}_E\,\Sigma = \mathcal{EPI} \wedge \mathrm{var}\ \Sigma = \mathrm{var}\,(\Sigma \cup \Delta)$ (here the symbol $\wedge$ denotes the meet of varieties). Clearly, if the class K_Σ is not a variety then $\mathrm{var}_E\,\Sigma$ contains some unary semigroups that are not epigroups. Moreover, the classes K_Σ and $\mathrm{var}_E\,\Sigma$ may differ even whenever K_Σ is a variety. This claim is confirmed by the following example that is communicated to the authors by V. Shaprynskiĭ.

Example 2.17. Let $\Sigma = \{x = x^2\}$. Consider the two-element semilattice $T = \{e, 0\}$. We define on T the unary operation * by the rule $e^* = 0^* = 0$. Results of the article [4] imply that any identity from Δ is strictly unary. Therefore, these identities hold in T, whence $T \in \mathcal{EPI} \wedge \mathrm{var}\ \Sigma = \mathrm{var}_E\,\Sigma$. But $\overline{e} = e$. Therefore, the unary operation * is not the pseudoinversion on T, thus $T \notin \mathbf{V}[\Sigma]$.

Recall that a semigroup identity $u = v$ is called *homotypical* if $c(u) = c(v)$, and *heterotypical* otherwise. The following claim gives a classification of all identity systems Σ such that $\mathbf{V}[\Sigma] = \mathrm{var}_E\,\Sigma$.

Lemma 2.18. *Let Σ be a system of identities written in the language of unary semigroups. The following are equivalent:*

a) $\mathbf{V}[\Sigma] = \mathrm{var}_E\,\Sigma$*;*
b) $\mathrm{var}_E\,\Sigma$ *satisfies a mixed identity;*
c) Σ *contains either a semigroup heterotypical identity or a mixed identity.*

Доказательство. a) $\longrightarrow$ c) Suppose that each identity in Σ is either homotypical or strictly unary. Obviously, the unary semigroup T from Example 2.17 satisfies all these identities, whence $T \in \mathrm{var}_E\,\Sigma$. But $T \notin \mathbf{V}[\Sigma]$, i. e. $\mathbf{V}[\Sigma] \neq \mathrm{var}_E\,\Sigma$.

c) $\longrightarrow$ b) If the identity $u = v$ is mixed then the required assertion is obvious. Suppose that the identity $u = v$ is heterotypical. We may assume that there is some letter x that occurs in the word u but does not occur in the word v. We substitute $\overline{x}$ to x in $u = v$. As a result, we obtain a mixed identity.

The implication b) $\longrightarrow$ a) follows from Corollary 2.14. $\square$

3. The proof of Theorem 1.1 and Corollary 1.2

The implication 4) $\longrightarrow$ 3) of Theorem 1.1 is obvious, while the implication 3) $\longrightarrow$ 2) follows from the evident fact that the variety $\mathcal{F}$ does not satisfy an identity of the form (1.4) with $\ell(w) > n$. It remains to verify the implications 1) $\longrightarrow$ 4) and 2) $\longrightarrow$ 1).

1) $\longrightarrow$ 4) Here we need some the following auxiliary fact.

Lemma 3.1. *Let* $\Sigma = \{p_\alpha = q_\alpha \mid \alpha \in \Lambda\}$. *If a variety* $\mathrm{var}_E\,\Sigma$ *satisfies an identity* $u = v$ *and* x *is a letter that does not occur in the words* p_α, q_α *(for all* $\alpha \in \Lambda$*),* u *and* v *then the identity* $ux = vx$ *follows from the identity system* $\Sigma' = \{p_\alpha x = q_\alpha x \mid \alpha \in \Lambda\}$ *in the class of all epigroups.*

Доказательство. In view of generally known universal algebraic facts, there exists a *deduction of the identity* $u = v$ *from the system of identities* $\Sigma \cup \Delta$, i. e. the sequence of identities

$$u_0 = v_0,\ u_1 = v_1,\ \ldots,\ u_m = v_m \tag{3.1}$$

such that the identity $u_0 = v_0$ lies in $\Sigma \cup \Delta$, the identity $u_m = v_m$ coincides with $u = v$ and, for each $i = 1, \ldots, m$, one of the following holds:

(i) the identity $u_i = v_i$ lies in $\Sigma \cup \Delta$;
(ii) there is $0 \le j < i$ such that $u_i \equiv v_j$ and $v_i \equiv u_j$;
(iii) there are $0 \le j, k < i$ such that $u_j \equiv u_i$, $v_j \equiv u_k$ and $v_k \equiv v_i$;
(iv) there are $0 \le j, k < i$ such that $u_i \equiv u_j u_k$ and $v_i \equiv v_j v_k$;
(v) there is $0 \le j < i$ such that $u_i \equiv \overline{u_j}$ and $v_i \equiv \overline{v_j}$;
(vi) there is $0 \le j < i$ such that the identity $u_i = v_i$ is obtained from the identity $u_j = v_j$ by a substitution of some word w for some letter that occurs in the identity $u_j = v_j$.

Let y be a letter with $y \not\equiv x$. If the letter x occurs in some identities of the sequence (3.1) then we substitute y to x in all such identities. The identities from $\Sigma \cup \{u = v\}$ will not change because these identities do not contain the letter x; and the identities from Δ will still remain in Δ. The sequence we obtain is a deduction of the identity $u = v$ from the identity system $\Sigma \cup \Delta$ again, and all the identities of this deduction do not contain the letter x. We may assume without any loss that already the deduction (3.1) possesses the last property.

For each $i = 0, 1, \ldots, m$, the identity $u_i = v_i$ holds in the variety $\mathrm{var}_E\,\Sigma$. Since the identity $u_m = v_m$ coincides with the identity $u = v$, it suffices to verify that, for each $i = 0, 1, \ldots, m$, the identity $u_i x = v_i x$ follows from the identity system

Σ' in the class of all epigroups. The proof of this claim is given by induction on i.

Induction base is evident because the identity $u_0 = v_0$ lies in $\Sigma \cup \Delta$.

Induction step. Let now $i > 0$. One can consider the cases (i)–(vi).

(i) This case is obvious.

(ii) By the induction assumption, the identity $u_j x = v_j x$ follows from the identity system Σ' in the class of all epigroups. Since the identity $u_i x = v_i x$ coincides with the identity $v_j x = u_j x$, we are done.

(iii) By the induction assumption, the identities $u_j x = v_j x$ (i. e. $u_i x = u_k x$) and $u_k x = v_k x$ (i. e. $u_k x = v_i x$) follow from the identity system Σ' in the class of all epigroups. Therefore, the identity $u_i x = v_i x$ follows from the identity system Σ' in the class of all epigroups too.

(iv) By the induction assumption, the identities $u_j x = v_j x$ and $u_k x = v_k x$ follow from the identity system Σ' in the class of all epigroups. We substitute $u_k x$ to x in the identity $u_j x = v_j x$. Since the letter x does not occur in the words u_j and v_j, we obtain the identity $u_j u_k x = v_j u_k x$, i. e. $u_i x = v_j u_k x$. Further, we multiply the identity $u_k x = v_k x$ on v_j from the left. Here we obtain the identity $v_j u_k x = v_j v_k x$, i. e. $v_j u_k x = v_i x$. We see that the identity system Σ' implies the identities $u_i x = v_j u_k x$ and $v_j u_k x = v_i x$ in the class of all epigroups, whence the identity $u_i x = v_i x$ also follows from Σ' in the class of all epigroups.

(v) By the induction assumption, the identity $u_j x = v_j x$ follows from the identity system Σ' in the class of all epigroups. Since $u_i \equiv \overline{u_j}$ and $v_i \equiv \overline{v_j}$, it remains to verify that the identity $\overline{u_j}\, x = \overline{v_j}\, x$ follows from the identity system Σ' in the class of all epigroups. Suppose that an epigroup S satisfies the identity $u_j x = v_j x$ and $\bigl|c(u_j) \cup c(v_j)\bigr| = k$. We fix arbitrary elements $a_1, \dots, a_k$ and b in S. Put $U_j = u_j(a_1, \dots, a_k)$ and $V_j = v_j(a_1, \dots, a_k)$. Then

$$U_j b = V_j b. \tag{3.2}$$

We need to verify that $\overline{U_j}\, b = \overline{V_j}\, b$. First of all, we verify that

$$V_j^{s+1} = U_j^s V_j \tag{3.3}$$

for any natural s. We use induction by s. If $s = 1$ then the equality (3.3) coincides with (3.2) where $b = V_j$. If $s > 1$ then

$$\begin{aligned} V_j^{s+1} &= V_j V_j^s \\ &= U_j V_j^s && \text{by (3.2) with } b = V_j^s \\ &= U_j U_j^{s-1} V_j && \text{by the inductive assumption} \\ &= U_j^s V_j, \end{aligned}$$

and the equality (3.3) is proved. The equality (3.2) with $b = V_j$ and (2.3) imply that $U_j^\omega V_j = \overline{U_j}\, U_j V_j = \overline{U_j}\, V_j^2$. Thus,

$$\overline{U_j}\, V_j^2 = U_j^\omega V_j. \tag{3.4}$$

Let now s be a natural number with $s \ge 2$. Using (3.4), we have

$$\overline{U_j}^{\,s} {V_j}^{s} = \overline{U_j}^{\,s-1} \bigl(\overline{U_j}\, {V_j}^{2}\bigr) {V_j}^{s-2} = \overline{U_j}^{\,s-1} {U_j}^{\omega} V_j {V_j}^{s-2} = \overline{U_j}^{\,s-1} {V_j}^{s-1}.$$

Therefore, $\overline{U_j}^{\,s} V_j^{\,s} = \overline{U_j}^{\,s-1} V_j^{\,s-1} = \cdots = \overline{U_j}\, V_j$. Thus,

$$\overline{U_j}^{\,s} V_j^{\,s} = \overline{U_j}\, V_j \tag{3.5}$$

for any natural s. Since S is an epigroup, there are numbers g and h such that $U_j^g, V_j^h \in \mathrm{Gr}\, S$. Put $m = \max\{g, h\}$. For any $s \ge m$ we have

$$\begin{aligned}
U_j^\omega V_j^s &= U_j^\omega (V_j^s V_j^\omega) && \text{because } V_j^s \in G_{V_j} \text{ by Lemma 2.7}\\
&= U_j^\omega (V_j^{s+1}\, \overline{V_j}\,) && \text{by (2.3)}\\
&= (U_j^\omega U_j^s) V_j\, \overline{V_j} && \text{by (3.3)}\\
&= (U_j^s V_j)\, \overline{V_j} && \text{because } U_j^s \in G_{U_j} \text{ by Lemma 2.7}\\
&= V_j^{s+1}\, \overline{V_j} && \text{by (3.3)}\\
&= V_j^s V_j^\omega && \text{by (2.3)}\\
&= V_j^s && \text{because } V_j^s \in G_{V_j} \text{ by Lemma 2.7.}
\end{aligned}$$

Thus,

$$U_j^\omega V_j^s = V_j^s \tag{3.6}$$

for any $s \ge m$. Note also that

$$\begin{aligned}
U_j^\omega V_j &= \overline{U_j}^{\,m} U_j^{\,m} V_j && \text{by (2.8)}\\
&= \overline{U_j^m}\, (U_j^m V_j) && \text{by (2.6)}\\
&= \overline{U_j^m}\, V_j^{m+1} && \text{by (3.3)}\\
&= \overline{U_j^m}\, V_j^{m+1} V_j^\omega && \text{because } V_j^{m+1} \in G_{V_j} \text{ by Lemma 2.7}\\
&= \overline{U_j^m}\, V_j^{m+1} (V_j^m)^\omega && \text{because } G_{V_j} = G_{V_j^m}\\
&= \overline{U_j^m}\, V_j^{m+1} V_j^m\, \overline{V_j^m} && \text{by (2.3)}\\
&= \big(\overline{U_j}^{\,m} V_j^{\,m}\big) V_j^{\,m+1}\, \overline{V_j^m} && \text{by (2.6)}\\
&= \overline{U_j}\, V_j V_j^{m+1}\, \overline{V_j^m} && \text{by (3.5)}\\
&= (\,\overline{U_j}\, V_j^2)(V_j^m\, \overline{V_j^m}\,) && \\
&= (U_j^\omega V_j)(V_j^m\, \overline{V_j^m}\,) && \text{by (3.4)}\\
&= V_j^{m+1}\, \overline{V_j^m} && \text{by (3.6)}\\
&= V_j V_j^{\,m}\, \overline{V_j}^{\,m} && \text{by (2.6)}\\
&= V_j V_j^\omega && \text{by (2.8)}\\
&= \overline{\overline{V_j}} && \text{by (2.4).}
\end{aligned}$$

Thus,

$$U_j^\omega V_j = \overline{\overline{V_j}}. \tag{3.7}$$

Besides that,

$$\begin{aligned}
\overline{U_j}\, V_j^\omega &= \big(\,\overline{U_j}\, V_j^{\,2}\big)\, \overline{V_j}^{\,2} && \text{by (2.8)}\\
&= (U_j^{\,\omega} V_j)\, \overline{V_j}^{\,2} && \text{by (3.4)}
\end{aligned}$$

$$\begin{aligned}
&= \overline{\overline{V_j}}\,\overline{V_j}^{\,2} && \text{by (3.7)}\\
&= V_j^\omega\,\overline{V_j} && \text{because } \overline{\overline{V_j}} \text{ and } \overline{V_j} \text{ are mutually inverse in } G_{V_j}\\
&= \overline{V_j} && \text{because } \overline{V_j} \in G_{V_j}.
\end{aligned}$$

Thus,

$$\overline{U_j}\,V_j^\omega = \overline{V_j}. \tag{3.8}$$

Finally, we have

$$\begin{aligned}
\overline{U_j}\,b &= \overline{U_j^2}\,(U_jb) && \text{by (2.5)}\\
&= \overline{U_j^2}\,(V_jb) && \text{by (3.2)}\\
&= \overline{U_j^2}\,(U_j^\omega V_j)b && \text{because } \overline{U_j^2} \in G_{U_j}\\
&= \overline{U_j^2}\,\big(\overline{\overline{V_j}}\,b\big) && \text{by (3.7)}\\
&= (\overline{U_j^2}\,V_j^\omega)V_jb && \text{by (2.4)}\\
&= \overline{U_j}^{\,2}V_j^{\,\omega}V_jb && \text{by (2.6)}\\
&= \overline{U_j}\,\overline{V_j}\,V_jb && \text{by (3.8)}\\
&= \overline{U_j}\,V_j^\omega\,\overline{V_j}\,V_jb && \text{because } \overline{V_j} \in G_{V_j}\\
&= \overline{V_j}\,(\overline{V_j}\,V_j)b && \text{by (3.8)}\\
&= \overline{V_j}\,V_j^\omega b && \text{by (2.3)}\\
&= \overline{V_j}\,b && \text{because } \overline{V_j} \in G_{V_j}.
\end{aligned}$$

We prove that $\overline{U_j}\,b = \overline{V_j}\,b$. This completes a consideration of the case (v).

(vi) By the induction assumption, the identity $u_jx = v_jx$ follows from the identity system Σ' in the class of all epigroups. We may assume without any loss that $c(u_j) \cup c(v_j) = \{x_1, \dots, x_k\}$. Since $x \notin c(u_i) \cup c(v_i)$, the letter x does not occur in the word w. We substitute w to x in the identity $u_jx = v_jx$. Then we obtain the identity $u_ix = v_ix$. Therefore, this identity follows from the identity system Σ' in the class of all epigroups.

Lemma is proved. □

Now we start with the direct proof of the implication 1) ⟶ 4). We are going to verify that if an epigroup variety $\mathcal{V}$ is a variety of degree $\le n$ then it satisfies an identity of the form (1.5) for some i and j with $1 \le i \le j \le n$. Clearly, this implies the desirable implication. We use induction by n.

Induction base. If $\mathcal{V}$ is a variety of degree 1 then it satisfies the identity of the form (1.5) with $i = j = n = 1$ by Lemma 2.6.

Induction step. Let $n > 1$ and $\mathcal{V}$ is a variety of degree $\le n$. If $\mathcal{P}, \overleftarrow{\mathcal{P}} \nsubseteq \mathcal{V}$ then $\mathcal{V}$ is a variety of epigroups with completely regular nth power by Lemma 2.12. By Lemma 2.6 $\mathcal{V}$ then satisfies the identity (2.9), i. e. the identity of the form (1.5) with $i = 1$ and $j = n$. Suppose now that $\mathcal{V}$ contains one of the varieties $\mathcal{P}$ or $\overleftarrow{\mathcal{P}}$. We will assume without loss of generality that $\mathcal{P} \subseteq \mathcal{V}$.

The variety $\mathcal{F}_{n+1}$ has degree $n+1$, whence $\mathcal{V} \not\supseteq \mathcal{F}_{n+1}$. Therefore, there is an identity $u = v$ that holds in $\mathcal{V}$ but is false in $\mathcal{F}_{n+1}$. In view of Lemma 2.5, every non-semigroup word equals to 0 in $\mathcal{F}_{n+1}$. It is evident that every non-linear semigroup word and every semigroup word of length $> n$ equal to 0 in $\mathcal{F}_{n+1}$ as well. Therefore, we may assume without any loss that u is a linear semigroup word of length $\le n$, i. e. $u \equiv x_1 \cdots x_m$ for some $m \le n$. Since $\mathcal{P} \subseteq \mathcal{V}$, the identity $x_1 \cdots x_m = v$ holds in $\mathcal{P}$. Now Lemma 2.10 successfully applies with the conclusion that $m > 1$ and $v \equiv v' x_m$ for some word v' with $c(v') = \{x_1, \dots, x_{m-1}\}$. Suppose that $\ell(v') \le m-1$. In particular, this means that v' is a semigroup word. Since $c(v') = \{x_1, \dots, x_{m-1}\}$, we have that $\ell(v') = m-1$. Therefore, the word v' is linear, whence v is linear too. This means that $u = v$ is a permutational identity. But every permutational identity holds in the variety $\mathcal{F}_{n+1}$, while the identity $u = v$ is false in $\mathcal{F}_{n+1}$. Hence $\ell(v') > m-1$.

Proposition 2.16 implies that the class of epigroups satisfying the identity

$$x_1 \cdots x_{m-1} = v' \tag{3.9}$$

is a variety. We denote this variety by $\mathcal{V}'$. According to Lemma 2.9, $\mathcal{V}'$ is a variety of degree $\le m-1$. Since $m \le n$, we use inductive assumption and conclude that $\mathcal{V}'$ satisfies the identity

$$x_1 \cdots x_{m-1} = x_1 \cdots x_{i-1} \cdot \overline{\overline{x_i \cdots x_j}} \cdot x_{j+1} \cdots x_{m-1}$$

for some $1 \le i \le j \le m-1$. Therefore, this identity follows from the identity $x_1 \cdots x_{m-1} = v'$ in the class of all epigroups. Further considerations are divided into two cases.

Case 1: the word v' contains the unary operation. According to Lemma 2.18, $\mathcal{V}' = \mathbf{V}[x_1 \cdots x_{m-1} = v'] = \mathrm{var}_E\{x_1 \cdots x_{m-1} = v'\}$. The letter x_m does not occur in any of the words

$$x_1 \cdots x_{m-1}, \ v' \ \text{and} \ x_1 \cdots x_{i-1} \cdot \overline{\overline{x_i \cdots x_j}} \cdot x_{j+1} \cdots x_{m-1}.$$

Now Lemma 3.1 successfully applies with the conclusion that the identity

$$x_1 \cdots x_m = x_1 \cdots x_{i-1} \cdot \overline{\overline{x_i \cdots x_j}} \cdot x_{j+1} \cdots x_m \tag{3.10}$$

follows in the class of all epigroups from $x_1 \cdots x_m = v' x_m$, i. e. from $x_1 \cdots x_m = v$. Therefore, $\mathcal{V}$ satisfies the identity (3.10). It is evident that this identity implies the identity (1.5).

Case 2: w' is a semigroup word. Substitute some letter x to all letters occurring in the identity (3.9). Then we obtain an identity $x^{m-1} = x^{m-1+k}$ for some $k > 0$. By (1.2), the latter identity implies in the class of all epigroups the identity $\overline{x} = x^{mk-1}$. Using Lemma 2.18 we have

$$\begin{aligned}\mathcal{V}' &= \mathbf{V}[x_1 \cdots x_{m-1} = v'] = \mathbf{V}[x_1 \cdots x_{m-1} = v', \overline{x} = x^{mk-1}]\\ &= \mathrm{var}_E\{x_1 \cdots x_{m-1} = v', \overline{x} = x^{mk-1}\}.\end{aligned}$$

Note that the variety $\mathbf{V}[x_1 \cdots x_m = v' x_m]$ satisfies the identity $x^m = x^{m+k}$. Hence, taking into account (1.3), we have that the identity $\overline{x} = x^{mk-1}$ holds in this variety. Then the variety $\mathbf{V}[x_1 \cdots x_m = v' x_m]$ satisfies the identity $\overline{x}\, x_m = x^{mk-1} x_m$. As in the Case 1, we apply Lemma 3.1. We get that the identity (3.10) holds in the variety $\mathbf{V}[x_1\cdots x_m = v'x_m]$. Then this variety satisfies the identity (1.5).

Thus, we complete the proof of the implication 1) $\longrightarrow$ 4).

2) $\longrightarrow$ 1). Let $\mathcal{V} \not\supseteq \mathcal{F}$. Then there is an identity $u = v$ that holds in $\mathcal{V}$ but does not hold in $\mathcal{F}$. Repeating literally arguments from the proof of the implication 1) $\longrightarrow$ 4), we reduce our consideration to the case when the word u is linear. Now Lemma 2.9 and the fact that every permutational identity holds in the variety $\mathcal{F}$ imply that $\mathcal{V}$ is a variety of finite degree.

Theorem 1.1 is proved. □

It remains to prove Corollary 1.2. The implication 1) $\longrightarrow$ 4) of this corollary follows from the proof of the same implication in Theorem 1.1. The implication 4) $\longrightarrow$ 3) is evident, while the implication 3) $\longrightarrow$ 2) follows from the evident fact that the variety $\mathcal{F}_{n+1}$ does not satisfy an identity of the form (1.4) with $\ell(w) > n$. Finally, the implication 2) $\longrightarrow$ 1) of Corollary 1.2 is verified quite analogously to the same implication of Theorem 1.1. □

Список литературы

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