SPECIAL ELEMENTS IN LATTICES
OF SEMIGROUP VARIETIES

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Аннотация. We survey results concerning special elements of eight types (namely, modular, lower-modular, upper-modular, distributive, codistributive, standard, costandard and neutral elements) in the lattice of all semigroup varieties and three of its sublattices, namely, the lattices of commutative varieties, of permutative varieties and of overcommutative ones. These results are due to Ježek, McKenzie, Shaprynskiĭ, Volkov and the author. No proofs are included but mainn ideas of proofs are briefly discussed. Several open questions are formulated.

1. INTRODUCTION

The collection of all semigroup varieties forms a lattice with respect to class-theoretical inclusion. This lattice will be denoted by **SEM**. The lattice **SEM** has been intensively studied since the beginning of 1960s. A systematic overview of the material accumulated here is given in the survey [19].

The lattice **SEM** has an extremely complicated structure. In particular, it contains an anti-isomorphic copy of the partition lattice over a countably infinite set [1,5], and therefore does not satisfy any non-trivial lattice identity. Identities in subvariety lattices of semigroup varieties were intensively examined in many articles. These articles contain a number of interesting and deep results (see [19, Section 11]). The next natural step is to consider varieties that guarantee, so to speak, 'nice lattice behaviour' in their neighborhood. Specifically, our attention is to study special elements of different types in the lattice **SEM**.

We will consider eight types of special elements: modular, lower-modular, upper-modular, distributive, codistributive, standard, costandard and neutral elements. Recall the corresponding definitions. An element x of a lattice $\langle L; \vee, \wedge \rangle$ is called

modular if	$\forall\, y, z \in L\colon$	$y \leq z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y;$
lower-modular if	$\forall\, y, z \in L\colon$	$x \leq y \longrightarrow x \vee (y \wedge z) = y \wedge (x \vee z);$

distributive if	$\forall\, y, z \in L\colon \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z);$
standard if	$\forall\, y, z \in L\colon \quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z);$

neutral if, for all $y, z \in L$, the sublattice of L generated by x, y and z is distributive. It is well known (see [4, Theorem 254 on p. 226], for instance) that an element $x \in L$ is neutral if and only if

$$\forall\, y, z \in L\colon \quad (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

Upper-modular, codistributive and *costandard* elements are defined dually to lower-modular, distributive and standard ones respectively.

Special elements play an important role in the general lattice theory (see [4, Section III.2], for instance). In particular, it is well known that if a is a neutral element in a lattice L then L is decomposable into a subdirect product of the principal ideal and the principal filter of L generated by a (see [4, Theorem 254 on p. 226], for instance). Thus the knowledge of which elements of a lattice are neutral gives essential information on the structure of the lattice as a whole.

There is a number of interrelations between types of elements we consider. It is evident that a neutral element is both standard and costandard; a standard or costandard element is modular; a [co]distributive element is lower-modular [upper-modular]. It is well known also that a [co]standard element is [co]distributive (see [4, Theorem 253 on p. 224], for instance). These interrelations between types of elements in abstract lattices are shown in Fig. 1.
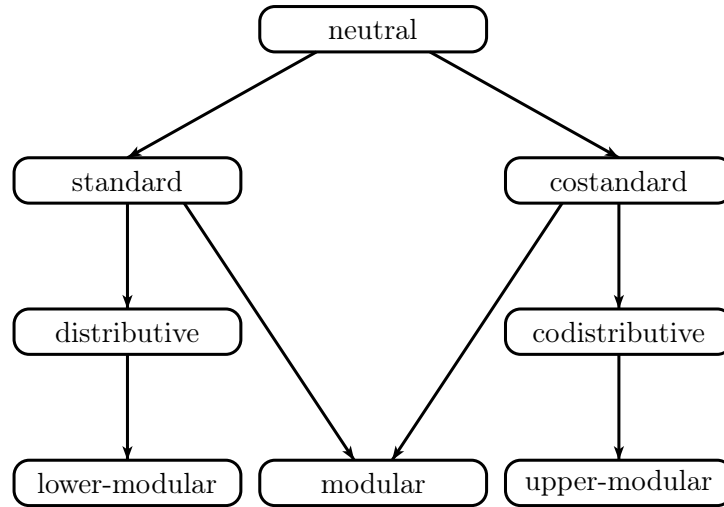
Рис. 1. Interrelations between types of elements in abstract lattices

First results about special elements in the lattice **SEM** were obtained in the articles [7, 31] where these results play an auxiliary role. A systematic examination of special elements in **SEM** is the objective of the articles [14, 15, 17, 22–27, 29, 32, 36]; see also [19, Section 14]. Briefly speaking, the mentioned articles contain complete descriptions of lower-modular, distributive, standard,

costandard and neutral elements of the lattice **SEM**[1] and essential information about modular, upper-modular and codistributive elements of this lattice (including strong necessary conditions and descriptions in wide and important partial cases). In particular, it turns out that there are some interrelations between special elements of different types in **SEM** that do not hold in abstract lattices. Namely, an element of **SEM** is standard if and only if it is distributive; is costandard if and only if it is neutral; is modular whenever it is lower-modular. Interrelations between types of elements in the lattice **SEM** are shown in Fig. 2. Note that there are no other interrelations between types of elements under consideration. Corresponding examples will be given below.
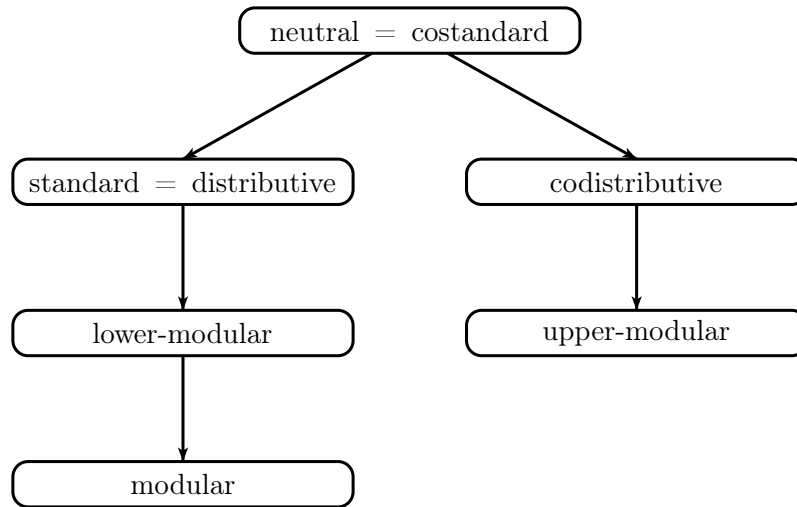

Рис. 2. Interrelations between types of elements in **SEM**

The lattice **SEM** contains a number of wide and important sublattices (see [19, Section 1 and Chapter 2]). It is natural to examine special elements in these sublattices. One of the most important sublattices of **SEM** is the lattice **Com** of all commutative semigroup varieties. It follows from results of [2] that this lattice contains an isomorphic copy of any finite lattice, and therefore does not satisfy any non-trivial lattice identity. On the other hand, the lattice **Com** is known to be countably infinite [10] and can be characterized [8] (see also [19, Section 8]). Special elements in the lattice **Com** are examined in [13, 14] where lower-modular, distributive, standard and neutral elements of **Com** are completely determined and an essential information about modular elements of this lattice is obtained. As in the case of the lattice **SEM**, it turns out that an element of **Com** is standard if and only if it is distributive; is modular whenever it is lower-modular. Interrelations between types of elements in the lattice **Com** are shown in Fig. 3. Two dotted arrows in this figure correspond to interrelations for which it is unknown whether they hold or not. No interrelations between

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[1]To prevent a possible confusion, we note that the description of standard elements of **SEM** is not formulated explicitly anywhere but readily follows from results of [29], see a comment to Theorem 3.3 below.

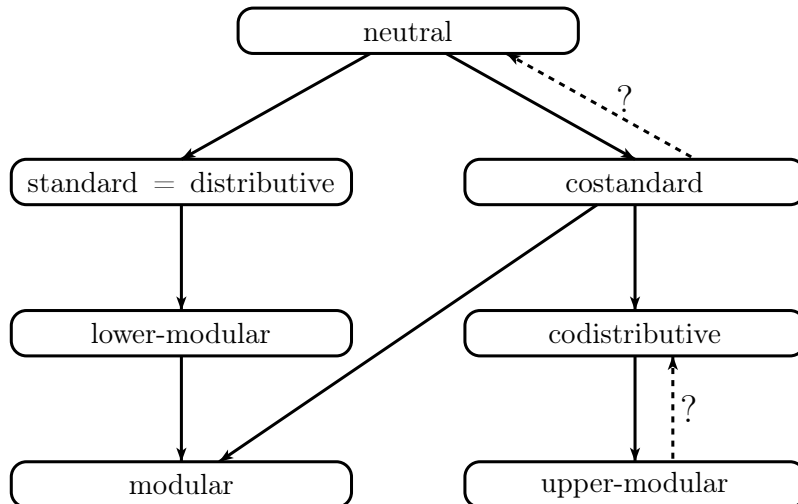types of elements in **Com** not specified in Fig. 3 hold. Corresponding examples will be given below.

Рис. 3. Interrelations between types of elements in **Com**

Recall that a semigroup variety is called *permutative* if it satisfies a *permutational* identity, that is, an identity of the type

$$(1) \qquad\qquad x_1 x_2 \cdots x_n = x_{1\pi} x_{2\pi} \cdots x_{n\pi}$$

where π is a non-trivial permutation on the set $\{1, 2, \ldots, n\}$. The collection of all permutative varieties forms a sublattice **Perm** of the lattice **SEM**. This lattice is located between **SEM** and **Com**. It seems quite natural to examine special elements in **Perm**. There are no published results here so far. Recently, Shaprynskiĭ has obtained some information about modular and lower-modular elements in the lattice **Perm**.

The 'antipode' of the lattice **Com** is the lattice **OC** of all *overcommutative* semigroup varieties (that is, varieties containing the variety of all commutative semigroups). It is well known that the lattice **SEM** is the disjoint union of **OC** and the lattice of all *periodic* semigroup varieties (that is, varieties consisting of periodic semigroups). Results of the papers [7, 23, 25] imply that if a semigroup variety $\mathcal{V}$ different from the variety of all semigroups belongs to one of the eight types mentioned above (with respect to **SEM**), then $\mathcal{V}$ is periodic (a somewhat more general fact is proved in [15], see Proposition 3.1 below). Thus an examination of special elements of all mentioned types in **SEM** a priori can not give any information about the lattice **OC**. Note that the lattice **OC** contains an isomorphic copy of any finite lattice [35], whence it does not satisfy any non-trivial lattice identity. Overcommutative varieties whose lattice of overcommutative subvarieties satisfies a particular lattice identity were intensively studied (see [19, Subsection 5.2] and the recent article [16]). All these arguments make the examination of special elements of **OC** very natural. Such an examination has been started in the article [21]. It is proved there that the properties of being a distributive, a codistributive, a standard, a costandard and a neutral element

of the lattice **OC** are equivalent, and a certain description of corresponding overcommutative varieties is proposed. But this description turns out to be incorrect (while the result that the five mentioned conditions are equivalent is true). The correct description of distributive, codistributive, standard, costandard and neutral elements of the lattice **OC** is contained in the article [18]. Interrelations between types of elements in the lattice **OC** are shown in Fig. 4.
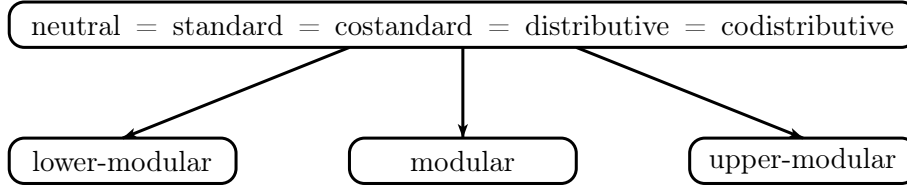

Рис. 4. Interrelations between types of elements in **OC**

This survey consists of six sections. In Section 2, we provide some preliminary results about special elements in abstract lattices, lattices of equivalence relations, congruence lattices of G-sets and the lattices **SEM** and **Com**. These preliminary results play an important role in the proofs of the results that we survey in Sections 3–6. In Sections 3 and 4, we overview results about special elements in the lattices **SEM** and **Com** respectively. Section 5 contains results about modular and lower-modular elements in lattices located between **SEM** and **Com**, namely in subvariety lattice of overcommutative varieties and in the lattice **Perm**. Sections 3–5 also contain several open questions. Finally, Section 6 is devoted to special elements in the lattice **OC**.

2. Preliminary results

2.1. ε-elements and Id-elements of lattices.
All types of special elements introduced above are defined by the same scheme. Namely, we take a particular identity and consider it as an open formula. Then, one of the variables is left free while all the others are subjected to a universal quantifier[2]. One can generalize this approach to an arbitrary lattice identity. This seems to be natural a priori and turns out to be quite fruitful a posteriori.

Let ε be a lattice identity of the form $s = t$ where terms s and t depend on an ordered set of variables $x_0, x_1, \ldots, x_n$. An element x of a lattice L is called an ε-*element* of L if

$$\forall\, x_1, \ldots, x_n \in L\colon \quad s(x, x_1, \ldots, x_n) = t(x, x_1, \ldots, x_n).$$

Note that we consider here two copies of the same identity with different orderings of its variables as distinct identities. An element of a lattice L is called an Id-*element* of L if it is an ε-element of L for some non-trivial identity ε.

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[2]Formally speaking, the definitions of modular, lower-modular and upper-modular elements are based on a lattice quasiidentity rather than an identity. But we give such definitions for the sake of brevity and convenience only. It is fairly easy to redefine these types of elements in the language of lattice identities.

For an element a of a lattice L, we put $(a] = \{x \in L \mid x \le a\}$. If $a \in L$ and the lattice $(a]$ satisfies the identity $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$ then

$$p(a \wedge x_1, \ldots, a \wedge x_n) = q(a \wedge x_1, \ldots, a \wedge x_n)$$

for all $x_1, \ldots, x_n \in L$ because $a \wedge x_1, \ldots, a \wedge x_n \in (a]$. Therefore, in this situation, a is an ε-element of L with the following identity ε:

$$p(x_0 \wedge x_1, \ldots, x_0 \wedge x_n) = q(x_0 \wedge x_1, \ldots, x_0 \wedge x_n).$$

So, we have the following

Observation 2.1. *If a is an element of a lattice L and the ideal $(a]$ of L satisfies some non-trivial lattice identity then a is an* Id*-element of L.*

The subvariety lattice of a variety $\mathcal{V}$ is denoted by $L(\mathcal{V})$. A semigroup variety $\mathcal{V}$ is called an Id-*variety* if it is an Id-element of the lattice **SEM**. The following assertion is a specialization of Observation 2.1 for the lattice **SEM**.

Corollary 2.2. *If $\mathcal{V}$ is a semigroup variety and the lattice $L(\mathcal{V})$ satisfies some non-trivial lattice identity then $\mathcal{V}$ is an* Id*-variety.*

The following fact turns out to be very helpful.

Proposition 2.3 ([13, Corollary 2.1])**.** *Let a be an atom and a neutral element of a lattice L and ε a lattice identity that holds in the 2-element lattice. An element $x \in L$ is an ε-element of the lattice L if and only if the element $x \vee a$ has the same property.*

We denote by $\operatorname{var} \Sigma$ the semigroup variety given by the identity system Σ. Put $\mathcal{SL} = \operatorname{var}\{x^2 = x,\ xy = yx\}$. It is well known that $\mathcal{SL}$ is an atom of the lattice **SEM** (see [19, Section 1], for instance) and a neutral element of this lattice (see [36, Proposition 4.1] or Theorem 3.4 below). Moreover, $\mathcal{SL}$ is a neutral atom of **Com**. Thus Proposition 2.3 implies the following

Corollary 2.4. *Let ε be a lattice identity that holds in the 2-element lattice. A [commutative] semigroup variety $\mathcal{V}$ is an ε-element of the lattice* **SEM** *[respectively* **Com***] if and only if the variety $\mathcal{V} \vee \mathcal{SL}$ has the same property.*

Note that a number of partial cases of Proposition 2.3 and Corollary 2.4 for special elements of different concrete types were proved earlier in [23, 27, 29, 32, 36].

2.2. Modular and upper-modular elements in lattices of equivalence relations.
If S is a set then $\operatorname{Eq}(S)$ stands for the lattice of equivalence relations on S.

Proposition 2.5. *Let S be a non-empty set. For an equivalence relation α on S, the following are equivalent:*

    a) *α is a modular element of the lattice $\operatorname{Eq}(S)$;*
    b) *α is an upper-modular element of the lattice $\operatorname{Eq}(S)$;*
    c) *α has at most one non-singleton class.*

The equivalence of the claims a) and c) of this proposition was proved in [6, Proposition 2.2], while the equivalence of the claims b) and c) was verified in [31, Proposition 3].

Proposition 2.5 turns out to be very helpful for the examination of modular and lower-modular elements in varietal lattices. In order to explain how this proposition can be applied, we need some new definitions and notation. Note that a semigroup S satisfies the identity system $wx = xw = w$ where the letter x does not occur in the word w if and only if S contains a zero element 0 and all values of w in S equal to 0. We adopt the usual convention of writing $w = 0$ as a short form of such a system and referring to the expression $w = 0$ as to a single identity. Identities of the form $w = 0$ are called 0-*reduced*. Further, let $\mathcal{X}$ be a semigroup variety, $\mathcal{V}$ a subvariety of $\mathcal{X}$, F the $\mathcal{X}$-free object and ν the fully invariant congruence on F corresponding to $\mathcal{V}$. It is clear that if $\mathcal{V}$ may be given within $\mathcal{X}$ by 0-reduced identities then ν has only one non-singleton class (the one that contains the equivalence classes modulo $\mathcal{X}$ that correspond to the left sides of those 0-reduced identities). Now recall the generally known fact that the lattice $L(\mathcal{X})$ is anti-isomorphic to the lattice of all fully invariant congruences on F. Therefore, the lattice $\mathrm{Eq}(F)$ contains an anti-isomorphic copy of $L(\mathcal{X})$. Combining all these observations with Proposition 2.5, we have the following

Corollary 2.6. *Let $\mathcal{X}$ be a semigroup variety and $\mathcal{V}$ its subvariety. If $\mathcal{V}$ is defined within $\mathcal{X}$ by 0-reduced identities then $\mathcal{V}$ is a modular and lower-modular element of the lattice $L(\mathcal{X})$.*

This statement permits to obtain an important information about modular and lower-modular elements of the lattices **SEM** and **Com** (see Subsections 3.2, 3.6, 4.1 and 4.4 below).

2.3. **Special elements in congruence lattices of G-sets.** A unary algebra with the carrier A and the set of (unary) operations G is called a *G-set* if G is equipped by a structure of a group and this group structure on G is compatible with the unary structure on A (this means that if $g, h \in G$, $x \in A$ and e is the unit element of G then $g\big(h(x)\big) = (gh)(x)$ and $e(x) = x$). Our interest to G-sets is explained by the fact that the lattice **OC** admits a concise and transparent description in terms of congruence lattices of G-sets. More precisely, **OC** is anti-isomorphic to a subdirect product of congruence lattices of countably infinite series of certain G-sets (see [35] or [19, Subsection 5.1]). To apply this result for examination of special elements in **OC**, some information about special elements in congruence lattices of G-sets is required.

A G-set A is said to be *transitive* if, for all $a, b \in A$, there exists $g \in G$ such that $g(a) = b$. If A is a G-set and $a \in A$ then we put

$$\mathrm{Stab}_A(a) = \big\{ g \in G \mid g(a) = a \big\}.$$

Clearly, $\mathrm{Stab}_A(a)$ is a subgroup in G. This subgroup is called a *stabilizer* of an element a in A. The congruence lattice of a G-set A is denoted by $\mathrm{Con}(A)$.

Proposition 2.7 ([21, Theorem 1]). *Let A be a non-transitive G-set such that $\mathrm{Stab}_A(x) = \mathrm{Stab}_A(y)$ for all elements $x, y \in A$. For a congruence α on A, the following are equivalent:*

a) α *is a distributive element of the lattice* $\mathrm{Con}(A)$;
b) α *is a codistributive element of the lattice* $\mathrm{Con}(A)$;
c) α *is a standard element of the lattice* $\mathrm{Con}(A)$;
d) α *is a costandard element of the lattice* $\mathrm{Con}(A)$;
e) α *is a neutral element of the lattice* $\mathrm{Con}(A)$;
f) α *is either the universal relation or the equality relation on* A.

G-sets that appear in [35] in the description of the lattice **OC** have the property that the stabilizer of any element in these G-sets is the trivial group. Thus, the application of Proposition 2.7 is not hindered by the hypothesis that stabilizers of all elements in A coincide. It is presently unknown if the proposition holds without this hypothesis.

2.4. **Upper-modular and codistributive elements: interrelations between lattice identities and a hereditary property.** The following easy observation turns out to be helpful.

Observation 2.8. *Let* L *be a lattice. If an element* $a \in L$ *is upper-modular [codistributive] in* L *and the lattice* $(a]$ *is modular [distributive] then every element of* $(a]$ *is upper-modular [codistributive] in* L.

This claim was noted in [25, Lemma 2.1] for upper-modular elements and in [27, Lemma 2.2] for codistributive ones.

Observation 2.8 immediately implies the following

Corollary 2.9. *Let* **Lat** *be one of the lattices* **SEM** *or* **Com**. *If a semigroup variety* $\mathcal{V}$ *is an upper-modular [codistributive] element of the lattice* **Lat** *and the lattice* $L(\mathcal{V})$ *is modular [distributive] then every subvariety of* $\mathcal{V}$ *is upper-modular [codistributive] element of the lattice* **Lat**.

3. The lattice **SEM**

For convenience, we call a semigroup variety *modular* if it is a modular element of the lattice **SEM** and adopt analogous convention for all other types of special elements. The main results of this section provide:

- a complete classification of lower-modular, distributive, standard, costandard or neutral varieties (Theorems 3.2, 3.3 and 3.4),
- a classification of modular, upper-modular or codistributive varieties in some wide partial cases (Theorems 3.10, 3.11, 3.18 and 3.26),
- strong necessary conditions for a semigroup variety to be modular, upper-modular or codistributive (Theorems 3.6, 3.7, 3.12 and 3.25),
- a sufficient condition for a semigroup variety to be modular (Theorem 3.8).

One can mention also Proposition 3.1 that gives an important information about Id-varieties.

3.1. **Id-varieties.** We denote by $\mathcal{SEM}$ the variety of all semigroups. A semigroup variety $\mathcal{V}$ is called *proper* if $\mathcal{V} \neq \mathcal{SEM}$.

The class of Id-varieties includes all varieties with non-trivial identities in subvariety lattices (see Corollary 2.2). It follows from results of [2] that a

semigroup variety $\mathcal{V}$ is periodic whenever the lattice $L(\mathcal{V})$ satisfies some non-trivial identity. As we have already mentioned in Section 1, results of the articles [7, 23, 25] imply that if a proper variety $\mathcal{V}$ belongs to one of the eight considered types in **SEM** then it is periodic too. All these statements are generalized by the following

Proposition 3.1 ([15, Theorem 1]). *A proper* Id-*variety of semigroups is periodic.*

On the other hand, it is verified in [15, Theorem 2] that there are periodic varieties (moreover, nil-varieties) that are not Id-varieties.

3.2. **Lower-modular varieties.** Varieties that may be given by 0-reduced identities only are called 0-*reduced*. We denote by $\mathcal{T}$ the trivial semigroup variety. A number of partial results concerning lower-modular varieties were obtained in [23, 24, 31]. All of them are covered by the following

Theorem 3.2. *A semigroup variety $\mathcal{V}$ is lower-modular if and only if either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a 0-reduced variety.*

This theorem was verified for the first time in [17, Theorem 1.1] and was reproved in a simpler and shorter way in [14]. The proof of Theorem 3.2 given in [14] is based on Theorem 5.1 below. Note that the 'if' part of Theorem 3.2 immediately follows from Corollaries 2.6 (with $\mathcal{X} = \mathcal{SEM}$) and 2.4.

Neutral, standard and distributive varieties are lower-modular. In view of Theorem 3.2, a description of varieties of these three types should look as follows:

A semigroup variety $\mathcal{V}$ is distributive [standard, neutral] if and only if either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a 0-reduced variety such that . . . (with some additional restriction to $\mathcal{N}$ depending on the type of element we consider).

Exact formulations of corresponding results are given in the following two subsections.

3.3. **Distributive and standard varieties.** Put

$$\mathcal{Q} = \mathrm{var}\{x^2 y = xyx = yx^2 = 0\},$$
$$\mathcal{Q}_n = \mathrm{var}\{x^2 y = xyx = yx^2 = x_1 x_2 \cdots x_n = 0\},$$
$$\mathcal{R} = \mathrm{var}\{x^2 = xyx = 0\},$$
$$\mathcal{R}_n = \mathrm{var}\{x^2 = xyx = x_1 x_2 \cdots x_n = 0\}$$

where n is a natural number. It is easy to see that varieties of these four types are precisely all 0-reduced varieties satisfying the identities $x^2 y = xyx = yx^2 = 0$.

Theorem 3.3. *For a semigroup variety $\mathcal{V}$, the following are equivalent:*

   a) *$\mathcal{V}$ is distributive;*
   b) *$\mathcal{V}$ is standard;*
   c) *either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is one of the varieties $\mathcal{Q}$, $\mathcal{Q}_n$, $\mathcal{R}$ or $\mathcal{R}_n$.*

The equivalence of the claims a) and c) of this theorem is proved in [29, Theorem 1.1]. Note that the proof of the implication a) $\longrightarrow$ c) given in [29] may be essentially simplified by using Theorem 3.2. The implication b) $\longrightarrow$ a) is evident. To verify the implication a) $\longrightarrow$ b), we need two ingredients. First, it is verified in [29, Corollary 1.2] that a distributive element of the lattice **SEM** is a modular element of this lattice[3]. Second, it is fairly easy to check that an element of a lattice is standard whenever it is both distributive and modular (see [13, Lemma 2.2], for instance). Note that the former statement is strengthened by Corollary 3.9 below.

3.4. **Costandard and neutral varieties.** Put $\mathcal{ZM} = \mathrm{var}\{xy = 0\}$. It is well known that $\mathcal{ZM}$ is an atom of the lattice **SEM**.

The following statement is a compilation of several published results.

Theorem 3.4. *For a semigroup variety $\mathcal{V}$, the following are equivalent:*
   a) *$\mathcal{V}$ is both lower-modular and upper-modular;*
   b) *$\mathcal{V}$ is both distributive and codistributive;*
   c) *$\mathcal{V}$ is costandard;*
   d) *$\mathcal{V}$ is neutral;*
   e) *$\mathcal{V}$ is one of the varieties $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$, $\mathcal{SL} \vee \mathcal{ZM}$ or $\mathcal{SEM}$.*

Clearly, the claim e) of this theorem may be reformulated in the manner specified in Subsection 3.2: either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a 0-reduced variety such that $xy = 0$ in $\mathcal{N}$.

We do not include in Theorem 3.4 the claim that $\mathcal{V}$ is both standard and costandard because it is well known that an element of arbitrary lattice is both standard and costandard if and only if it is neutral (see [4, Theorem 255 on p. 228], for instance). The equivalence of the claims a) and e) of Theorem 3.4 was verified in [24, Corollary 3.5], the equivalence of c) and e) was checked in [27, Theorem 1.3], the equivalence of d) and e) was proved in [36, Proposition 4.1], while the implications d) $\longrightarrow$ b) $\longrightarrow$ a) are evident.

Since a neutral element of a lattice is standard, Theorem 3.4 implies the following

Corollary 3.5 ([27, Corollary 1.1]). *Every costandard semigroup variety is standard.*

3.5. **An application to definable varieties.** Here we discuss an interesting application of results overviewed above. We need some new definitions. A subset A of a lattice $\langle L; \vee, \wedge \rangle$ is called *definable in L* if there exists a first-order formula $\Phi(x)$ with one free variable x in the language of lattice operations $\vee$ and $\wedge$ which *defines A in L*. This means that, for an element $a \in L$, the sentence $\Phi(a)$ is true if and only if $a \in A$. If A consists of a single element, then we talk about definability of this element. A set X of semigroup varieties (or a single semigroup variety $\mathcal{X}$) is said to be *definable* if it is definable in **SEM**. In this situation we will say that the corresponding first-order formula *defines* the set X or the variety $\mathcal{X}$.

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[3]This immediately follows from the implication a) $\longrightarrow$ c) of Theorem 3.3 and Corollaries 2.6 (with $\mathcal{X} = \mathcal{SEM}$) and 2.4.

A number of deep results about definable varieties and sets of varieties of semigroups have been obtained in [7] by Ježek and McKenzie[4]. It has been conjectured there that every finitely based semigroup variety is definable up to duality. The conjecture is confirmed in [7] for locally finite finitely based varieties. On their way to obtaining this fundamental result, Ježek and McKenzie proved the definability of several important sets of semigroup varieties such as the sets of all finitely based, all locally finite, all finitely generated and all 0-reduced semigroup varieties. But the article [7] contains no explicit first-order formulas that define any of these sets of varieties. The task of writing an explicit formula that defines the set of all finitely based or the set of all locally finite or the set of all finitely generated varieties seems to be extremely difficult. On the other hand, the set of all 0-reduced varieties can be defined by a quite simple first-order formula based on descriptions of lower-modular and neutral varieties.

Indeed, Theorem 3.2 shows that a semigroup variety is 0-reduced if and only if it is lower-modular and does not contain the variety $\mathcal{SL}$. It remains to define the variety $\mathcal{SL}$. Theorem 3.4 together with the well-known description of atoms of the lattice **SEM** (see [19, Section 1], for instance) imply that this lattice contains exactly two neutral atoms, namely the varieties $\mathcal{SL}$ and $\mathcal{ZM}$. Recall that a semigroup variety $\mathcal{V}$ is called *chain* if the lattice $L(\mathcal{V})$ is a chain. It is well known that the variety $\mathcal{ZM}$ is properly contained in some chain variety, while the variety $\mathcal{SL}$ is not (see [28], for instance, for more details). Combining the mentioned observations, we see that the class of all 0-reduced varieties may be defined as the class **K** of semigroup varieties with the following properties:

   (i) every member of **K** is a lower-modular variety;
   (ii) if $\mathcal{V} \in \mathbf{K}$ and $\mathcal{V}$ contains some neutral atom $\mathcal{A}$ then $\mathcal{A}$ is properly contained in some chain variety.

It is evident that properties (i) and (ii) may be written by simple first-order formulas with one free variable.

An explicit formula that defines the class of all 0-reduced varieties is written in [28, Section 3]. Note that the description of distributive semigroup varieties given by Theorem 3.3 may also be applied to define some interesting varieties (see [28, Section 6]).

3.6. **Modular varieties.** The problem of description of modular semigroup varieties is open so far. Here we provide some partial results concerning this problem.

Recall that a semigroup variety is called a *nil-variety* if it consists of nilsemigroups or, equivalently, satisfies an identity of the form $x^n = 0$ for some natural n. Clearly, every 0-reduced variety is a nil-variety. The following theorem gives a strong necessary condition for a semigroup variety to be modular.

Theorem 3.6. *If $\mathcal{V}$ is a modular semigroup variety then either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a nil-variety.*

---

[4]Note that paper [7] deals with the lattice of equational theories of semigroups, that is, the dual of **SEM** rather than the lattice **SEM** itself. When reproducing results from [7], we adapt them to the terminology of the present article.

This theorem readily follows from [7, Proposition 1.6]. A deduction of Theorem 3.6 from [7, Proposition 1.6] is given explicitly in [22, Proposition 2.1]. A direct and transparent proof of Theorem 3.6 is given in [14]. This proof is based on Theorem 5.1 below.

Theorem 3.6 and Corollary 2.4 completely reduce the examination of modular varieties to nil-varieties. There is a strong necessary condition for a nil-variety to be modular. To formulate this result, we need some additional definitions.

We call an identity $u = v$ *substitutive* if the words u and v depend on the same letters and v may be obtained from u by renaming of letters. In [6], Ježek describes modular elements of the lattice of all varieties (more precisely, all equational theories) of any given type. In particular, it follows from [6, Lemma 6.3] that if a nil-variety of semigroups $\mathcal{V}$ is a modular element of the lattice of all groupoid varieties then $\mathcal{V}$ may be given by 0-reduced and substitutive identities only. This does not imply directly the same conclusion for modular nil-varieties because a modular element of **SEM** need not be a modular element of the lattice of all groupoid varieties. Nevertheless, the following assertion shows that the 'semigroup analogue' of the mentioned result of Ježek holds true.

Theorem 3.7 ([22, Proposition 2.2]). *A modular nil-variety of semigroups may be given by* 0-*reduced and substitutive identities only.*

Corollary 2.6 with $\mathcal{X} = \mathcal{SEM}$ immediately implies the following

Theorem 3.8. *Every* 0-*reduced semigroup variety is modular.*

This fact was noted for the first time in [31, Corollary 3] and rediscovered (in different terminology) in [7, Proposition 1.1].

Theorems 3.7 and 3.8 provide a necessary and a sufficient condition for a nil-variety to be modular respectively. The gap between these conditions seems to be not very large. But the necessary condition is not a sufficient one, while the sufficient condition is not a necessary one (this follows from Theorem 3.10 below).

Theorems 3.2 and 3.8 and Corollary 2.4 immediately imply the following

Corollary 3.9 ([17, Corollary 1.2]). *Every lower-modular semigroup variety is modular.*

By the way, neither of the five other possible interrelations between properties of being a modular variety, a lower-modular variety and an upper-modular variety holds. For instance:

- the variety $\mathrm{var}\{x^2 = 0,\ xy = yx\}$ is modular (by Theorem 3.10 below) but not lower-modular (by Theorem 3.2);
- the variety $\mathrm{var}\{xyz = 0\}$ is modular and lower-modular (by Corollary 2.6 with $\mathcal{X} = \mathcal{SEM}$) but not upper-modular (by Theorem 3.11 below);
- an arbirary abelian periodic group variety is upper-modular (by Theorem 3.11 below) but neither modular nor lower-modular (by Theorems 3.6 and 3.2 respectively).

Theorems 3.7 and 3.8 show that in order to describe modular nil-varieties (and therefore all modular varieties) we need to examine nil-varieties satisfying

substitutive identities. A natural partial case of substitutive identities are permutational ones, while the strongest permutational identity is the commutative law. Modular varieties satisfying this law are completely classified by the following

Theorem 3.10 ([22, Theorem 3.1]). *A commutative semigroup variety $\mathcal{V}$ is modular if and only if $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ satisfies the identity*

$$(2) \qquad\qquad\qquad\qquad x^2 y = 0.$$

3.7. **Upper-modular varieties.** The problem of description of upper-modular semigroup varieties is open so far. Here we provide some partial results concerning this problem. The first result classifies upper-modular varieties in some wide class of varieties. To formulate this statement we need some additional definitions and notation.

A semigroup variety $\mathcal{V}$ is called a variety of *finite degree* [a variety of *degree n*] if all nilsemigroups in $\mathcal{V}$ are nilpotent [if nilpotency degrees of nilsemigroups in $\mathcal{V}$ are bounded by the number n and n is the least number with this property]. We say that a semigroup variety is a *variety of degree $> n$* if it is either a variety of a finite degree m with $m > n$ or not a variety of finite degree. Put

$$\mathcal{A}_n = \mathrm{var}\{x^n y = y,\ xy = yx\} \ \text{ where }\ n \geq 1,$$
$$\mathcal{C} = \mathrm{var}\{x^2 = x^3,\ xy = yx\}.$$

In particular, $\mathcal{A}_1 = \mathcal{T}$. Note that $\mathcal{A}_n$ is the variety of all Abelian groups whose exponent divides n.

Theorem 3.11 ([26, Theorem 1]). *A semigroup variety $\mathcal{V}$ of degree > 2 is upper-modular if and only if one of the following holds:*
   (i) $\mathcal{V} = \mathcal{SEM}$;
   (ii) $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ *where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a nil-variety satisfying the identities $x^2 y = xy^2$ and $xy = yx$;*
   (iii) $\mathcal{V} = \mathcal{A}_n \vee \mathcal{M} \vee \mathcal{N}$ *where $n \geq 1$, $\mathcal{M}$ is one of the varieties $\mathcal{T}$, $\mathcal{SL}$ or $\mathcal{C}$ and $\mathcal{N}$ is a commutative variety satisfying the identity (2).*

We note that Theorem 3.11 readily implies a necessary condition for a semigroup variety to be upper-modular given by [25, Theorem 1.1] and a description of upper-modular nil-varieties obtained in [32, Theorem 2].

Theorem 3.11 reduces the examination of upper-modular varieties to varieties of degree ≤ 2. To formulate a result concerning this case, we need some new definitions and notation. Recall that a semigroup variety is called *completely regular* if it consists of *completely regular* semigroups — unions of groups. A semigroup variety $\mathcal{V}$ is called a *variety of semigroups with completely regular square* if, for any member S of $\mathcal{V}$, the semigroup S^2 is completely regular. Put

$$\mathcal{LZ} = \mathrm{var}\{xy = x\},$$
$$\mathcal{RZ} = \mathrm{var}\{xy = y\},$$
$$\mathcal{P} = \mathrm{var}\{xy = x^2 y,\ x^2 y^2 = y^2 x^2\},$$
$$\overleftarrow{\mathcal{P}} = \mathrm{var}\{xy = xy^2,\ x^2 y^2 = y^2 x^2\}.$$

All we know about upper-modular varieties of degree ≤ 2 is the following

Theorem 3.12 ([26, Theorem 2]). *If $\mathcal{V}$ is an upper-modular semigroup variety of degree ≤ 2 then one of the following holds:*

    (i) *$\mathcal{V}$ is a variety of semigroups with completely regular square;*

    (ii) *$\mathcal{V} = \mathcal{K} \vee \mathcal{P}$ where $\mathcal{K}$ is a completely regular semigroup variety such that $\mathcal{RZ} \nsubseteq \mathcal{K}$;*

    (iii) *$\mathcal{V} = \mathcal{K} \vee \overleftarrow{\mathcal{P}}$ where $\mathcal{K}$ is a completely regular semigroup variety such that $\mathcal{LZ} \nsubseteq \mathcal{K}$.*

We do not know any example of a non-upper-modular variety that satisfies one of the claims (i)–(iii) of Theorem 3.12. This inspires the following two questions.

Question 3.13. *Is it true that every variety of semigroups with completely regular square is upper-modular?*

Question 3.14. *Is it true that every semigroup variety satisfying one of the claims* (ii) *or* (iii) *of Theorem 3.12 is upper-modular?*

A natural weaker version of Question 3.13 is the following

Question 3.15. *Is it true that every completely regular semigroup variety is upper-modular?*

Although Theorems 3.11 and 3.12 do not provide a classification of all upper-modular varieties, they permit the deduction of some important and surprising properties of such varieties. Theorems 3.11 and 3.12, together with results of the articles [34, 37], imply the following

Corollary 3.16 ([26, Corollary 2]). *A proper upper-modular semigroup variety has a modular subvariety lattice.*

Corollaries 3.16 and 2.9 imply the following

Corollary 3.17 ([26, Corollary 3]). *If a proper semigroup variety is upper-modular then every its subvariety is also upper-modular.*

Now we describe upper-modular varieties in one more class of varieties. A semigroup variety is called *strongly permutative* if it satisfies an identity of the form (1) with $1\pi \neq 1$ and $n\pi \neq n$.

Theorem 3.18. *A strongly permutative semigroup variety $\mathcal{V}$ is upper-modular if and only if it satisfies one of the claims* (ii) *or* (iii) *of Theorem* 3.11.

A partial case of this statement concerning commutative varieties is proved in [25, Theorem 1.2]. Theorem 3.18 may be easily deduced from the proof of this partial case. A scheme of this deduction is provided in [24].

As we have seen above (see Corollary 3.16), the subvariety lattice of arbitrary proper upper-modular variety is modular. It turns out that such a lattice is even distributive in several wide classes of varieties. So, Theorem 3.11, together with results of the paper [33], implies the following

Corollary 3.19 ([26, Corollary 1]). *A proper upper-modular semigroup variety of degree > 2 has a distributive subvariety lattice.*

Theorem 3.18, together with results of [33], implies the following

Corollary 3.20. *A strongly permutative upper-modular semigroup variety has a distributive subvariety lattice.*

The special case of this claim dealing with commutative varieties was mentioned in [25, Corollary 4.4].

Theorem 3.12, together with results of the articles [11] and [34], readily implies the following

Corollary 3.21 ([26, Corollary 4]). *Let $\mathcal{V}$ be a proper upper-modular semigroup variety that is not a variety of semigroups with completely regular square and let ε be a non-trivial lattice identity. The lattice $L(\mathcal{V})$ satisfies the identity ε (in particular, is distributive) if and only if the subvariety lattice of any group subvariety of $\mathcal{V}$ has the same property.*

Further, a semigroup variety $\mathcal{V}$ is called *combinatorial* if all groups in $\mathcal{V}$ are trivial. Corollary 3.21, together with the result of the paper [3], readily implies the following

Corollary 3.22 ([26, Corollary 5]). *A combinatorial upper-modular semigroup variety has a distributive subvariety lattice.*

Corollaries 3.19–3.22 inspire the following open

Question 3.23. *Is it true that the subvariety lattice of every proper upper-modular semigroup variety is distributive?*

All proper upper-modular varieties that appeared above are varieties mentioned in Theorem 3.11. These varieties are commutative. Based on this observation, one can conjecture that any proper upper-modular variety is commutative. But this is not the case. Evident counter-examples are the varieties $\mathcal{LZ}$ and $\mathcal{RZ}$. The claim that these two varieties are upper-modular immediately follows from the fact that they are atoms of the lattice **SEM**. Two more examples of proper non-commutative upper-modular varieties are the varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$. Indeed, it is well known that if a variety $\mathcal{V}$ is properly contained in one of these two varieties then $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{ZM}$, whence $\mathcal{V}$ is lower-modular by Theorem 3.2. This readily implies that $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$ are upper-modular.

3.8. **Varieties that are both modular and upper-modular.** It is interesting to examine varieties that satisfy different combinations of the properties we consider. Corollary 3.9 implies that a variety is both modular and lower-modular if and only if it is lower-modular. So, Theorem 3.2 gives, in fact, a complete description of varieties that are both modular and lower-modular (this result was obtained for the first time in [36, Theorem 3.1]). A description of varieties that both are lower-modular and upper-modular as well as varieties that are both distributive and codistributive is given in Theorem 3.4. The following assertion classifies varieties that are both modular and upper-modular.

Proposition 3.24 ([32, Theorem 1]). *A semigroup variety $\mathcal{V}$ is both modular and upper-modular if and only if either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a commutative variety satisfying the identity* (2).

3.9. **Codistributive varieties.** The problem of description of codistributive semigroup varieties is open so far. Here we provide some partial results concerning this problem. The following theorem gives a strong necessary condition for a semigroup variety to be codistributive.

Theorem 3.25 ([27, Theorem 1.1]). *If a semigroup variety $\mathcal{V}$ is codistributive then either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V}$ is a variety of semigroups with completely regular square.*

Note that Theorems 3.11 and 3.12 are crucial in the proof of Theorem 3.25.

It is easy to see that a variety of semigroups with completely regular square is a variety of degree ≤ 2 (this readily follows from [12, Lemma 1] or [25, Proposition 2.11]). Therefore, Theorem 3.25 implies that a proper codistributive variety has degree ≤ 2. The following assertion shows that, for strongly permutative varieties, the converse statement holds as well.

Theorem 3.26 ([27, Theorem 1.2]). *For a strongly permutative semigroup variety $\mathcal{V}$, the following are equivalent:*

  a) *$\mathcal{V}$ is codistributive;*
  b) *$\mathcal{V}$ is a variety of degree ≤ 2;*
  c) *$\mathcal{V} = \mathcal{A}_n \vee \mathcal{X}$ where $n \geq 1$ and $\mathcal{X}$ is one of the varieties $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$ or $\mathcal{SL} \vee \mathcal{ZM}$.*

Clearly, every costandard variety is codistributive, while every codistributive variety is upper-modular. But the reverse statements do not hold. For instance:

  • the variety $\mathcal{A}_n$ with $n > 1$ is codistributive (by Theorem 3.26) but not costandard (by Theorem 3.4);
  • the variety $\mathcal{C}$ is upper-modular (by Theorem 3.11) but not codistributive (by Theorem 3.25).

It is easy to see that there exist non-codistributive varieties of semigroups with completely regular square and moreover, non-codistributive periodic group varieties. Indeed, the lattice of periodic group varieties is modular but not distributive. Therefore it contains the 5-element modular non-distributive sublattice. It is evident that all three pairwise non-comparable elements of this sublattice are non-codistributive periodic group varieties. We see that the problem of description of codistributive varieties is closely related to the problem of description of periodic group varieties with distributive subvariety lattice. The latter problem seems to be extremely difficult (see [19, Subsection 11.2] for more detailed comments), whence the former problem is extremely difficult too.

However, we do not know any examples of non-codistributive varieties of semigroups with completely regular square except ones mentioned in the previous paragraph. This inspires us to eliminate an examination of codistributive varieties with non-trivial groups. In other words, it seems natural to consider combinatorial codistributive varieties only. It is easy to see that if $\mathcal{V}$ is a combinatorial variety of semigroups with completely regular square then, for every $S \in \mathcal{V}$, the semigroups S^2 is a band. A variety with the last property is called a *variety of semigroups with idempotent square*. In view of Theorem 3.25, every combinatorial codistributive variety is a variety of semigroups with idempotent square. Thus the following question seems to be natural.

Question 3.27. *Is it true that an arbitrary variety of semigroups with idempotent square is codistributive?*

A natural weaker version of this question is the following

Question 3.28. *Is it true that an arbitrary variety of bands is codistributive?*

Clearly, every variety of semigroups with idempotent square satisfies the identity $xy = (xy)^2$. Put

$$\mathcal{IS} = \mathrm{var}\{xy = (xy)^2\},$$
$$\mathcal{BAND} = \mathrm{var}\{x = x^2\}.$$

It is verified in [3] that the lattice $L(\mathcal{IS})$ is distributive. Then Corollary 2.9 shows that Question 3.27 is equivalent to the following: is the variety $\mathcal{IS}$ codistributive? Analogously, Question 3.28 is equivalent to asking whether the variety $\mathcal{BAND}$ is codistributive or not, that is, whether the equality

$$\mathcal{BAND} \wedge (\mathcal{X} \vee \mathcal{Y}) = (\mathcal{BAND} \wedge \mathcal{X}) \vee (\mathcal{BAND} \wedge \mathcal{Y})$$

holds for arbitrary varieties $\mathcal{X}$ and $\mathcal{Y}$ or not. It is verified in [9, Corollary 5.9] that this is the case whenever the varieties $\mathcal{X}$ and $\mathcal{Y}$ are locally finite.

A strongly permutative codistributive variety has a distribitive subvariety lattice (this follows from Corollary 3.20 and may be easily deduced from Theorem 3.26). Combinatorial codistributive varieties also have a distribitive subvariety lattice (here it suffices to refer to either Corollary 3.22 or Theorem 3.25 and the mentioned result of [3]). We do not know any example of proper codistributive variety with non-distributive subvariety lattice. This inspires the following

Question 3.29. *Is it true that the subvariety lattice of an arbitrary proper codistributive semigroup variety is distributive?*

This question is closely related to the following

Question 3.30. *Is it true that every subvariety of an arbitrary proper codistributive semigroup variety is codistributive?*

Corollary 2.9 shows that the affirmative answer to Question 3.29 would imply the affirmative answer to Question 3.30.

All proper codistributive varieties appeared above are varieties mentioned in Theorem 3.26. These varieties are commutative. Based on this observation, one can conjecture that any proper codistributive variety is commutative. But this is not the case. To provide a corresponding example, we formulate the following

Remark 3.31 ([27, Remark 4.1]). *If $\mathcal{V}_1$, $\mathcal{V}_2$, ..., $\mathcal{V}_k$ are atoms of the lattice* **SEM** *then the variety* $\bigvee_{i=1}^{k} \mathcal{V}_i$ *is codistributive.*

In particular, non-commutative varieties $\mathcal{LZ}$ and $\mathcal{RZ}$ are codistributive. In connection with Questions 3.29 and 3.30, we note that if $\mathcal{V}_1$, $\mathcal{V}_2$, ..., $\mathcal{V}_k$ are atoms of the lattice **SEM** and $\mathcal{V} = \bigvee_{i=1}^{k} \mathcal{V}_i$ then:

(i) the lattice $L(\mathcal{V})$ is distributive (in fact, $L(\mathcal{V})$ is a direct product of k copies of 2-element chains),

(ii) if $\mathcal{X} \subseteq \mathcal{V}$ then $\mathcal{X}$ is the join of those of the atoms $\mathcal{V}_1$, $\mathcal{V}_2$, ..., $\mathcal{V}_k$ that are contained in $\mathcal{V}$, and therefore $\mathcal{X}$ is codistributive by Remark 3.31.

The claim (i) is a part of [30, Proposition 1], while the statement (ii) follows from (i).

4. The lattice **Com**

For convenience, we call a commutative semigroup variety **Com**-*modular* if it is a modular element of the lattice **Com** and adopt analogous convention for all other types of special elements. The main results of this section provide:

- a complete classification of **Com**-lower-modular, **Com**-distributive, **Com**-standard or **Com**-neutral varieties (Theorems 4.1, 4.2 and 4.3),
- necessary conditions for a commutative semigroup variety to be **Com**-modular (Theorems 4.6 and 4.7),
- a sufficient condition for a commutative semigroup variety to be **Com**-modular (Theorem 4.8).

4.1. **Com-lower-modular varieties.** We denote by $\mathcal{COM}$ the variety of all commutative semigroups. A commutative semigroup variety is called **Com**-0-*reduced* if it may be given by the commutative law and some non-empty set of 0-reduced identities only. Some partial information about **Com**-lower-modular varieties was obtained in [13]. It is covered by the following 'commutative analogue' of Theorem 3.2.

Theorem 4.1 ([14, Theorem 1.6]). *A commutative semigroup variety $\mathcal{V}$ is **Com**-lower-modular if and only if either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a **Com**-0-reduced variety.*

Note that the 'if' part of Theorem 4.1 immediately follows from Corollaries 2.6 (with $\mathcal{X} = \mathcal{COM}$) and 2.4. The proof of the 'only if' part given in [14] is based on Theorem 5.1 below.

As in the case of the lattice **SEM** (see Subsection 3.2), Theorem 4.1 implies that a description of **Com**-distributive, **Com**-standard and **Com**-neutral varieties should look as follows:

*A commutative semigroup variety $\mathcal{V}$ is **Com**-distributive [**Com**-standard, **Com**-neutral] if and only if either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a **Com**-0-reduced variety such that ...* (with some additional restriction to $\mathcal{N}$ depending on the type of element we consider).

Exact formulations of corresponding results are given in the following two subsections.

4.2. **Com-distributive and Com-standard varieties.** The following statement is the 'commutative analogue' of Theorem 3.3.

Theorem 4.2 ([13, Theorem 1.1]). *For a commutative semigroup variety $\mathcal{V}$, the following are equivalent:*

a) *$\mathcal{V}$ is **Com**-distributive;*
b) *$\mathcal{V}$ is **Com**-standard;*

c) *either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a **Com**-0-reduced variety that satisfies the identities $x^3yz = x^2y^2z = 0$ and either satisfies both the identities $x^3 = 0$ and $x^2y^2 = 0$ or does not satisfy any of them.*

It is verified in [13, Corollary 1.1] that a **Com**-distributive variety is **Com**-modular. This statement also follows from Corollary 4.9 below.

4.3. **Com-neutral varieties.** A complete description of **Com**-neutral varieties is given by the following partial analogue of Theorem 3.4.

Theorem 4.3. *For a commutative semigroup variety $\mathcal{V}$, the following are equivalent:*
   a) *$\mathcal{V}$ is both **Com**-upper-modular and **Com**-lower-modular;*
   b) *$\mathcal{V}$ is both **Com**-distributive and **Com**-codistributive;*
   c) *$\mathcal{V}$ is **Com**-neutral;*
   d) *either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and the variety $\mathcal{N}$ satisfies the identity (2).*

The equivalence of the claims b)–d) of this theorem is verified in [13, Theorem 1.2], while the equivalence of the claims a) and c) is proved in [14, Corollary 4.2].

Theorem 4.3, together with results of [33], implies the following

Corollary 4.4. *If $\mathcal{V}$ is a **Com**-neutral commutative semigroup variety and $\mathcal{V} \neq \mathcal{COM}$ then the lattice $L(\mathcal{V})$ is distributive.*

Theorem 4.3 also implies the following

Corollary 4.5. *If $\mathcal{V}$ is a **Com**-neutral commutative semigroup variety and $\mathcal{V} \neq \mathcal{COM}$ then every subvariety of $\mathcal{V}$ is **Com**-neutral.*

It is interesting to compare Theorems 4.3 and 3.10. We see that a commutative semigroup variety $\mathcal{V}$ with $\mathcal{V} \neq \mathcal{COM}$ is **Com**-neutral if and only if it is modular.

4.4. **Com-modular varieties.** The problem of description of **Com**-modular semigroup varieties is open so far. Here we provide some partial results concerning this problem. Note that these results are 'commutative analogues' of Theorems 3.6, 3.7 and 3.8.

First of all, the following necessary condition for a commutative semigroup variety to be **Com**-modular is true.

Theorem 4.6 ([14, Theorem 1.4]). *If $\mathcal{V}$ is a **Com**-modular commutative semigroup variety then either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a nil-variety.*

In fact, this theorem readily follows from Theorem 5.1 below. Theorem 4.6 and Corollary 2.4 completely reduce an examination of **Com**-modular varieties to the nil-case. The following theorem is yet another analogue of the result of Ježek [6] (see Theorem 3.7 and the paragraph before this theorem).

Theorem 4.7 ([14, Theorem 1.5]). *A **Com**-modular commutative nil-variety of semigroups may be given within the variety $\mathcal{COM}$ by 0-reduced and substitutive identities only.*

Corollary 2.6 with $\mathcal{X} = \mathcal{COM}$ immediately implies the following

Theorem 4.8 ([13, Proposition 2.1]). *Every* **Com**-0-*reduced commutative semigroup variety is* **Com**-*modular.*

Theorems 4.7 and 4.8 provide, respectively, a necessary and a sufficient condition for a commutative nil-variety to be **Com**-modular. The gap between these conditions does not seem to be very large. But the necessary condition is not a sufficient one, while the sufficient condition is not a necessary one. Indeed, it may be checked that:

- the variety $\mathrm{var}\{xyzt = x^3 = 0,\, x^2y = y^2x,\, xy = yx\}$ is **Com**-modular although it is not **Com**-0-reduced,
- the variety $\mathrm{var}\{x^5 = 0,\, x^3y^2 = y^3x^2,\, xy = yx\}$ is not **Com**-modular although it is given within $\mathcal{COM}$ by 0-reduced and substitutive identities only

(Shaprynskiĭ, private communication).

Theorems 4.1 and 4.8 and Corollary 2.4 imply the following 'commutative analogue' of Corollary 3.9.

Corollary 4.9 ([14, Corollary 4.1]). *Every* **Com**-*lower-modular commutative semigroup variety is* **Com**-*modular.*

We note that neither of the five other possible interrelations between properties of being a **Com**-modular, a **Com**-lower-modular and a **Com**-upper-modular variety holds. For instance:

- the variety $\mathrm{var}\{xyzt = x^3 = 0,\, x^2y = y^2x,\, xy = yx\}$ is **Com**-modular (as we have already mentioned above) but not **Com**-lower-modular (by Theorem 4.1);
- the variety $\mathrm{var}\{x^3 = 0,\, xy = yx\}$ is **Com**-modular and **Com**-lower-modular (by Corollary 2.6 with $\mathcal{X} = \mathcal{COM}$) but not **Com**-upper-modular (by Proposition 4.10 below);
- the variety $\mathcal{A}_p$ with p prime is **Com**-upper-modular (because this variety is an atom of **Com**) but neither **Com**-modular nor **Com**-lower-modular (by Theorems 4.6 and 4.1 respectively).

4.5. **Com-upper-modular, Com-codistributive and Com-costandard varieties.** The problems of description of these three types of varieties are open so far. The only partial result here is the following

Proposition 4.10. *For a* **Com**-0-*rediced commutative semigroup variety* $\mathcal{V}$, *the following are equivalent:*

   a) $\mathcal{V}$ *is* **Com**-*upper-modular;*
   b) $\mathcal{V}$ *is* **Com**-*codistributive;*
   c) $\mathcal{V}$ *is* **Com**-*costandard;*
   d) $\mathcal{V}$ *is* **Com**-*neutral;*
   e) $\mathcal{V}$ *satisfies the identity* (2).

The implication a) $\longrightarrow$ e) of this proposition (as well as the reverse implication) is verified in [13, Proposition 2.3], the implication e) $\longrightarrow$ d) follows from Theorem 4.3, while the implications d) $\longrightarrow$ c) $\longrightarrow$ b) $\longrightarrow$ a) are evident.

At the conclusion of Section 4, we discuss interrelations between properties to be a **Com**-neutral, a **Com**-costandard and a **Com**-upper-modular variety. It is easy to see that there exist **Com**-codistributive but not **Com**-costandard varieties. Indeed, the variety $\mathcal{A}_p$ with a prime p is codistributive by Remark 3.31, and moreover is **Com**-codistributive. But this variety is not **Com**-modular by Theorem 4.6, whence it is not **Com**-costandard. The following two questions are open so far.

Question 4.11. *Is it true that an arbitrary* **Com**-*costandard commutative semigroup variety is* **Com**-*neutral?*

Question 4.12. *Is it true that an arbitrary* **Com**-*upper-modular commutative semigroup variety is* **Com**-*codistributive?*

5. Lattices located between **SEM** and **Com**

In this section, we examine modular and lower-modular elements only. It turns out that properties of such elements in the lattices **SEM** and **Com** discussed in Subsections 3.2, 3.6, 4.1 and 4.4 may be partially extended to some sublattices of **SEM** that contain **Com**. More precisely, we have in mind subvariety lattices of overcommutative semigroup varieties and the lattice **Perm**.

5.1. **Subvariety lattices of overcommutative varieties.** As we have seen in Subsections 4.1 and 4.4, there are numerous parallels between results about modular and lower-modular elements in the lattices **SEM** and **Com**. The following result partially explains these parallels and permits us to give unified proofs of several results about [lower-]modular elements in **SEM** and **Com**.

Theorem 5.1 ([14, Proposition 3.3]). *Let $\mathcal{X}$ be an overcommutative semigroup variety and $\mathcal{V}$ a periodic subvariety of $\mathcal{X}$. If $\mathcal{V}$ is either a modular or a lower-modular element of the lattice $L(\mathcal{X})$ then $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a nil-variety.*

Applying this theorem with $\mathcal{X} = \mathcal{SEM}$ [respectively $\mathcal{X} = \mathcal{COM}$], we obtain an important information about [**Com**-]modular and [**Com**-]lower-modular varieties. After that, only some simple additional arguments are needed to verify Theorems 3.8 and 4.8, as well as the 'if' parts of Theorems 3.2 and 4.1. One can speculate if it is possible to eliminate these additional arguments altogether. To do this, we should verify an analogue of Theorem 5.1 without the assumption that the variety $\mathcal{V}$ is periodic. Unfortunately, it turns out that this is impossible. Indeed, it is verified in [20] that every proper semigroup variety is covered in **SEM** by some other variety (see also [19, Subsection 3.1]). It is evident that if an overcommutative variety $\mathcal{V}$ is covered by a variety $\mathcal{X}$ then $\mathcal{X}$ is overcommutative and $\mathcal{V}$ is a lower-modular element of the lattice $L(\mathcal{X})$. Thus, the 'lower-modular half' of Theorem 5.1 would be false if we eliminate the assumption that $\mathcal{V}$ is periodic. The same is true for the 'modular half' of this theorem. For example, the variety $\mathcal{COM}$ is a modular element in the lattice $L(\mathcal{W})$ where $\mathcal{W} = \text{var}\{xyz = yzx = zyx\}$ (Shaprynskiĭ, private communication). Note that $\mathcal{COM}$ is also a lower-modular element in $L(\mathcal{W})$ because $\mathcal{W}$ covers $\mathcal{COM}$.

5.2. **The lattice Perm.** By analogy with the commutative case, we call a permutative semigroup variety **Perm**-[*lower-*]*modular* if it is a [lower-]modular element of the lattice **Perm**. The following assertion is proved recently by Shaprynskiĭ (unpublished).

Theorem 5.2. *If a permutative semigroup variety* $\mathcal{V}$ *is either* **Perm**-*modular or* **Perm**-*lower-modular then* $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ *where* $\mathcal{M}$ *is one of the varieties* $\mathcal{T}$ *or* $\mathcal{SL}$ *and* $\mathcal{N}$ *is a nil-variety.*

This result does not give any information about **Perm**-modular or **Perm**-lower-modular nil-varieties. Recall that:

  (i) by Theorems 3.7 and 4.7, every [**Com**-]modular nil-variety may be given [within **Com**] by substitutive and 0-reduced identities only;
 (ii) by Theorems 3.2 and 4.1, every [**Com**-]lower-modular nil-variety is [**Com**-]0-reduced;
(iii) by Corollary 2.6, every [**Com**-]0-reduced variety is both [**Com**-]modular and [**Com**-]lower-modular.

Note that we cannot use Corollary 2.6 to obtain a 'permutational analogue' of the claim (iii) because the class of all permutative semigroups does not form a variety.

We do not know whether a 'permutational analogue' of the claim (i) true. So, we formulate the following

Question 5.3. *Is it true that an arbitrary* **Perm**-*modular permutative nil-variety of semigroups may be given by substitutive and* 0-*reduced identities only?*

As to 'permutational analogues' of claims (ii) and (iii), they do not hold. For instance:

  • the variety var$\{xyzt = 0,\ x^2y = xyx\}$ is **Perm**-lower-modular although it may not be given by permutational and 0-reduced identities only;
  • the variety var$\{x_1x_2x_3x_4x_5 = 0,\ xy = yx\}$ is neither **Perm**-modular nor **Perm**-lower-modular although it is permutative and is given by permutational and 0-reduced identities only

(Shaprynskiĭ, private communication).

6. THE LATTICE **OC**

For convenience, we call an overcommutative semigroup variety **OC**-*modular* if it is a modular element of the lattice **OC** and adopt analogous convention for all other types of special elements.

The problems of description of **OC**-modular, **OC**-lower-modular and **OC**-upper-modular varieties are open so far. Moreover, any essential information about varieties of these three types is absent. On the other hand, **OC**-distributive, **OC**-codistributive, **OC**-standard, **OC**-costandard and **OC**-neutral varieties are completely determined. To formulate their description, we need some new definitions and notation.

Let m and n be positive integers with $2 \leq m \leq n$. A sequence of positive integers $(\ell_1, \ell_2, \ldots, \ell_m)$ is called a *partition of n into m parts* if

$$\sum_{i=1}^{m} \ell_i = n \quad \text{and} \quad \ell_1 \geq \ell_2 \geq \cdots \geq \ell_m.$$

The set of all partitions of n into m parts is denoted by $\Lambda_{n,m}$. Let $\lambda = (\ell_1, \ell_2, \ldots, \ell_m) \in \Lambda_{n,m}$. We define numbers $q(\lambda)$, $r(\lambda)$ and $s(\lambda)$ as follows:

$q(\lambda)$ is the number of ℓ_i's with $\ell_i = 1$;

$r(\lambda) = n - q(\lambda)$ (in other words, $r(\lambda)$ is the sum of all ℓ_i's with $\ell_i > 1$);

$s(\lambda) = \max\{r(\lambda) - q(\lambda) - \delta, 0\}$ where

$$\delta = \begin{cases} 0 & \text{if } n = 3, \, m = 2 \text{ and } \lambda = (2,1), \\ 1 & \text{otherwise.} \end{cases}$$

If $k \geq 0$ then $\lambda^{(k)}$ stands for the following partition of $n + k$ into $m + k$ parts:

$$\lambda^{(k)} = (\ell_1, \ell_2, \ldots, \ell_m, \underbrace{1, \ldots, 1}_{k \text{ times}})$$

(in particular, $\lambda^{(0)} = \lambda$). If $\mu = (m_1, m_2, \ldots, m_s) \in \Lambda_{r,s}$ then $W_{r,s,\mu}$ stands for the set of all words u such that:

- the length of u equals r;
- u depends on the letters $x_1, x_2, \ldots, x_s$;
- for every $i = 1, 2, \ldots, s$, the number of occurrences of x_i in u equals m_i.

For a partition $\lambda = (\ell_1, \ell_2, \ldots, \ell_m) \in \Lambda_{n,m}$, we put

$$\mathcal{S}_\lambda = \text{var}\{u = v \mid \text{there is } i \in \{0, 1, \ldots, s(\lambda)\} \text{ such that } u, v \in W_{n+i, m+i, \lambda^{(i)}}\}.$$

We call sets of the form $W_{n,m,\lambda}$ *transversals*. We say that an overcommutative variety $\mathcal{V}$ *reduces* [*collapses*] a transversal $W_{n,m,\lambda}$ if $\mathcal{V}$ satisfies some non-trivial identity [all identities] of the form $u = v$ with $u, v \in W_{n,m,\lambda}$. An overcommutative variety $\mathcal{V}$ is said to be *greedy* if it collapses any transversal it reduces.

Theorem 6.1. *For an overcommutative semigroup variety $\mathcal{V}$, the following are equivalent:*

a) $\mathcal{V}$ *is* **OC***-distributive;*

b) $\mathcal{V}$ *is* **OC***-codistributive;*

c) $\mathcal{V}$ *is* **OC***-standard;*

d) $\mathcal{V}$ *is* **OC***-costandard;*

e) $\mathcal{V}$ *is* **OC***-neutral;*

f) $\mathcal{V}$ *is greedy;*

g) *either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V} = \bigwedge_{i=1}^{k} \mathcal{S}_{\lambda_i}$ for some partitions $\lambda_1, \lambda_2, \ldots, \lambda_k$.*

The equivalence of claims a)–f) of this theorem was proved in [21] (claim f) was not mentioned in [21] explicitly but the fact that this claim is equivalent to each of the claims a)–e) readily follows from the proofs in [21]). The results of paper [35] and Proposition 2.7 play crucial role in this part of the proof of Theorem 6.1. The equivalence of claims f) and g) is verified in [18].

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