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ON REACHABILITY ANALYSIS FOR NONLINEAR CONTROL SYSTEMS WITH STATE CONSTRAINTS

Mikhail Gusev

N.N.Krasovskii Institute of Mathematics and Mechanics
S.Kovalevskaya str., 16
620099, Ekaterinburg, Russia
Ural Federal University
620002, Ekaterinburg, Russia

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Abstract. The paper is devoted to the problem of approximating reachable sets of a nonlinear control system with state constraints given as a solution set for a nonlinear inequality. A procedure to remove state constraints is proposed; this procedure consists in replacing a primary system by an auxiliary system without state constraints. The equations of the auxiliary system depend on a small parameter. It is shown that a reachable set of the primary system may be approximated in the Hausdorff metric by reachable sets of the auxiliary system when the small parameter tends to zero. The estimates of the rate of convergence are given.

1. Introduction.

This paper describes the algorithm for computing the reachable sets of a control system with state constraints which are given as a level set of a continuously differentiable function. The proposed algorithm is based on removing state constraints by replacing the original system with an auxiliary system, which is obtained by modifying the set of velocities of the original system outside the constraints. The right-hand side of this system is dependent on a small parameter. The reachable sets found for the auxiliary system without state constraints, approximate in the Hausdorff metric the reachable set of the original system with state constraints as the small parameter tends to zero.

Different approaches for computing the reachable sets, including those for systems with state constraints, are presented in [14, 17, 16, 6, 13, 1, 3, 11, 7, 8]. The method of removing state constraints in the construction of reachable sets for differential inclusions was proposed in [12], where the tube of trajectories of the differential inclusion with convex state constraint was approximated by the solutions of a family of differential inclusions without constraints depending on a matrix "penalty" parameter. In the paper [9] we removed state constraints restricting velocities of the original system near the constraint border. The right-hand side of the approximating system depends on a scalar penalty, and the reachable sets of

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this system tends from the inside to the reachable set of the original system with state constraints when the penalty tends to infinity.

In this paper we propose another procedure for removing state constraints. This procedure is based on construction of an auxiliary control system without state constraints. The right-hand side of this auxiliary system depends on a small scalar parameter, it's reachable set contains the reachable set of the original system with state constraints. Together with the results of [9] it allows to find two-sided estimates for reachable sets. We prove the convergence of reachable sets of auxiliary systems in the Hausdorff metric to the reachable set of the original system when the small parameter tends to zero. The estimate of the rate of convergence is given also.

2. **Problem statement.** Consider the control system

$$\dot{x}(t) = f(x(t), u(t)),\ t_0 \le t \le \theta, \quad x(t_0) = x^0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is a state vector and $u(t)$ is a control. The constraints on the control have the form

$$u(t) \in U,\ a.e.\ t \in [t_0, \theta], \tag{2}$$

where U is a compact set in $\mathbb{R}^r$, the controls are measurable function $u : [t_0, \theta] \to U$, we use $\mathcal{U}$ to denote the set of controls.

Further we use the following notation. By $A^\top$ we denote the transpose of a real matrix A, 0 stands for a zero vector of appropriate dimension. For $x, y \in \mathbb{R}^n$ let $(x, y) = x^\top y$ be an inner product, $x^\top = (x_1, \ldots, x_n)$, $\|x\| = (x, x)^{\frac{1}{2}}$ be an Euclidean norm, and $B_r(\bar{x})$: $B_r(\bar{x}) = \{x \in \mathbb{R}^n : \|x - \bar{x}\| \le r\}$ be a ball of radius $r > 0$ centered at $\bar{x}$. For a set $S \subset \mathbb{R}^n$ let ∂S, $\mathrm{int} S$, $\mathrm{cl} S$, $\mathrm{co} S$ be a boundary, an interior, a closure, and a convex hull of S respectively; $\nabla g(x)$ is the gradient of a function $g(x)$ at the point x, $h(A, B)$ is the Hausdorff distance between two sets $A, B \subset \mathbb{R}^n$, $\mathrm{conv}(\mathbb{R}^n)$ denotes a family of compact convex subsets of $\mathbb{R}^n$.

Further we suppose the following

Assumption 1. *The mapping $f(x, u) : \mathbb{R}^n \times U \to \mathbb{R}^n$ satisfies the conditions*

1) $f(x, u)$ is continuous in x, u and locally Lipschitz in x uniformly in $u \in U$;
2) linear growth condition: there exists $C > 0$ such that

$$\|f(x, u)\| \le C(1 + \|x\|),\ (x, u) \in \mathbb{R}^n \times U;$$

3) the set $F(x) := f(x, U)$ is convex for all x.

The control system (1) is equivalent to the differential inclusion

$$\dot{x}(t) \in F(x(t)), \quad x(t_0) = x^0, \tag{3}$$

where $F : \mathbb{R}^n \to \mathrm{conv}(\mathbb{R}^n)$ is locally Lipschitz with respect to the Hausdorff distance. Let $x(t, u(\cdot), x^0)$ denote the solution of system (1)— an absolutely continuous function satisfying (1) a.e. and the initial condition $x(t_0) = x^0$.

The state constraints are given by the inclusion

$$x(t) \in S,\ t \in [t_0, \theta], \tag{4}$$

where S is a level set of a continuously differentiable function $g : \mathbb{R}^n \to \mathbb{R}$:

$$S = \{x \in \mathbb{R}^n : g(x) \le 0\}. \tag{5}$$

The reachable set of the system (1) with state constraints (4) at time θ is the set

$$G_0(\theta) = \{x \in \mathbb{R}^n : \exists u(\cdot) \in \mathcal{U},\ x = x(\theta, u(\cdot), x^0),\ x(t, u(\cdot), x^0) \in S,\ t_0 \le t \le \theta\}.$$

This is the set of all points to which the system (1) can be moved at time θ under constraints (2), (4). By $G(\theta)$ we denote the reachable set of system (1) without state constraints

$$G(\theta) = \{x \in \mathbb{R}^n : \exists u(\cdot) \in \mathcal{U},\ x = x(\theta, u(\cdot), x^0)\}.$$

Assumption 1 implies that $G(\theta)$ is a compact set in $\mathbb{R}^n$ and all trajectories of (1) with initial state $x(t_0) = x^0$ are contained in a certain ball $B_R(\bar{x})$, we will denote this ball as B_R.

We consider here the following problem: to construct the control system

$$\dot{x}(t) = f_\varepsilon(x(t), u(t)),\ x(t_0) = x^0, \tag{6}$$

depending on a small parameter $\varepsilon > 0$ such that

1) the mapping $f_\varepsilon(x, u)$ is defined for x from a certain neighbourhood of $S \cap B_R$ and for u from U; $f_\varepsilon(x, u)$ is continuous in x, u and locally Lipschitz in x uniformly in $u \in U$;
2) $f_\varepsilon(x, U) \subset f(x, U)$, $f_\varepsilon(x, U) = f(x, U)$ for $x \in S \cap B_R$;
3) $G_\varepsilon(\theta) \to G_0(\theta)$ in the Hausdorff metric as $\varepsilon \to 0$, where $G_\varepsilon(\theta)$ is the reachable set of system (6) without state constraints.

Thus, the original control system is substituted by a family of control systems without state constraints those reachable sets approximate the reachable set of the original system.

We say that system (6) is an approximating system for (1).

Further constructions are based on the following inward pointing condition (see, e.g., [4, 5, 2, 19]).

Assumption 2. *For all x in $\{x \in \mathbb{R}^n : g(x) = 0\} \cap B_R$*

$$\min_{u \in U} (\nabla g(x), f(x, u)) < 0. \tag{7}$$

This condition provides that the reachable set $G_0(\theta)$ is nonempty.

Proposition 1. *If assumption 2 is fulfilled, then there exists a positive number σ such that inequality (7) holds for all points of the set*

$$S_R^\sigma = \{x \in \mathbb{R}^n : 0 \le g(x) \le \sigma\} \cap B_R.$$

Proof. From (7) it follows that $\nabla g(x) \ne 0$ for $x \in \{x \in \mathbb{R}^n : g(x) = 0\} \cap B_R$. Since the set $\{x \in \mathbb{R}^n : g(x) = 0\} \cap B_R$ is compact and the function

$$\eta(x) = \min_{u \in U} (\nabla g(x), f(x, u))$$

is continuous, there exists $\delta > 0$ such that inequality (7) holds for all x from the δ-neighborhood of the set $\{x \in \mathbb{R}^n : g(x) = 0\} \cap B_R$. Since $\nabla g(x) \ne 0$ at the points of this δ-neighborhood, there exists $K > 0$ such that

$$d(x) \le K|g(x)|,$$

where $d(x)$ is the distance from x to the boundary of S [7]. To complete the proof, take $\sigma = \delta/K$. □

Further, we will use the following extension of inward pointing condition.

Assumption 3. *There exist a positive number σ and a feedback control $\bar{u} : S_R^\sigma \to U$ such that the function $f(x, \bar{u}(x))$ is Lipschitz continuous on S_R^σ and*

$$(\nabla g(x), f(x, \bar{u}(x))) < 0, \quad \forall x \in S_R^\sigma. \tag{8}$$

Assuming the latter to hold we define the right-hand side $f_\varepsilon(x,u)$ of system (6) on the set $\{x \in \mathbb{R}^n : g(x) \le \sigma\} \cap B_R$ as follows. Take $0 < \varepsilon < \sigma$. Let $h_\varepsilon(\tau) : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $0 \le h_\varepsilon(\tau) \le 1$, $h_\varepsilon(\tau) = 1$ if $\tau < 0$, $h_\varepsilon(\tau) = 0$ if $\tau > \varepsilon$. Define

$$f_\varepsilon(x,u) = \begin{cases} h_\varepsilon(g(x))f(x,u) + (1 - h_\varepsilon(g(x)))f(x,\bar{u}(x)) & \text{if} \quad g(x) > 0, \\ f(x,u) & \text{if} \quad g(x) \le 0. \end{cases}$$

We can take as $h_\varepsilon(\tau)$, for example, a linear-quadratic function

$$h_\varepsilon(\tau) = \begin{cases} 1 & \text{if} \quad \tau < 0, \\ 1 - a\tau^2 & \text{if} \quad 0 \le \tau \le d\varepsilon, \\ 1 - a(d\varepsilon)^2 - b(\tau - d\varepsilon) & \text{if} \quad d\varepsilon < \tau < (1-d)\varepsilon, \\ a(\tau - \varepsilon)^2 & \text{if} \quad (1-d)\varepsilon \le \tau \le \varepsilon, \\ 0 & \text{if} \quad \tau > \varepsilon, \end{cases} \tag{9}$$

where $0 < d < 1/2$, $a = 1/(2d(1-d)\varepsilon^2)$, $b = 1/((1-d)\varepsilon)$.

Theorem 2.1. *Let the function $f(x,u)$ and the constraints of the problem satisfy assumptions 1, 3. Then*

1) *for $0 < \varepsilon < \sigma$ the mapping $f_\varepsilon(x,u)$ is continuous on $\{x \in \mathbb{R}^n : g(x) \le \sigma\} \cap B_R \times U$ and Lipschitz continuous in x uniformly in $u \in U$;*
2) *for any $u(\cdot) \in \mathcal{U}$ the solution $x_\varepsilon(t)$ of system (6) with initial data $x_\varepsilon(t_0) = x^0$ is defined on $[t_0, \theta]$ and satisfies the inequality*
$$g(x_\varepsilon(t)) \le \varepsilon, \quad t \in [t_0, \theta]; \tag{10}$$
3) *for all ε, $0 < \varepsilon < \sigma$, the inclusion $G_0(\theta) \subset G_\varepsilon(\theta)$ holds true. There exists $L > 0$ such that*
$$h(G_0(\theta), G_\varepsilon(\theta)) \le L\varepsilon. \tag{11}$$

Proof. The function $f_\varepsilon(x,u)$ coincides with $f(x,u)$ on the set $S_1 \times U$ where $S_1 = \{x : g(x) \le 0\} \cap B_R$, hence it is continuous. For $(x,u) \in S_2 \times U$, $S_2 = \{x : 0 \le g(x) \le \sigma\} \cap B_R$, $f_\varepsilon(x,u)$ is continuous as a superposition of continuous functions. The continuity of $f_\varepsilon(x,u)$ at the points (x,u), where $g(x) = 0$, may be proved using standard arguments. For the proof of the Lipschitz condition for $f_\varepsilon(x,u)$ we note first that there exist constants $L_1, L_2 > 0$, independent on u such that for $i = 1,2$

$$|f_\varepsilon(x,u) - f_\varepsilon(y,u)| \le L_i\|x - y\|, \quad \forall x,y \in S_i, \ \forall u \in U.$$

Here L_2 depends obviously on ε. Denote $L_3 = \max\{L_1, L_2\}$. Let $x \in S_1$, $y \in S_2$, connect the points x, y by a line interval. At the end points of the interval the values of g have the opposite signs. Hence, there exists a point z from this interval such that $g(z) = 0$. Since $z \in S_i, i = 1,2$, we have

$$|f_\varepsilon(x,u) - f_\varepsilon(y,u)| \le |f_\varepsilon(x,u) - f_\varepsilon(z,u)| + |f_\varepsilon(z,u) - f_\varepsilon(y,u)|$$
$$\le L_1\|x - z\| + L_2\|y - z\| \le L_3(\|x - z\| + \|y - z\|) = L_3\|x - y\|,$$

for all $u \in U$.

Consider the solution $x_\varepsilon(t)$ of system (6) corresponding to the control $u(\cdot) \in \mathcal{U}$. Since $f_\varepsilon(x,u)$ is a convex combination of the vectors $f(x,u)$ and $f(x,\bar{u}(x))$ belonging to the convex set $f(x,U)$, we see that $\dot{x}_\varepsilon(t) \in f(x_\varepsilon(t), U)$ a.e. . From Filippov's lemma (see [10]) it follows that there exists a control $u_\varepsilon(\cdot) \in \mathcal{U}$ such that

$$\dot{x}_\varepsilon(t) = f(x_\varepsilon(t), u_\varepsilon(t)) \ a.e..$$

This means that every admissible trajectory of the auxiliary system is the admissible trajectory of system (1). We show that this trajectory lies in the set $\{x \in \mathbb{R}^n : g(x) \leq \sigma\} \cap B_R$. Let γ^* be the maximum of the numbers γ not exceeding θ such that $x_\varepsilon(t)$ is defined on the interval $[t_0, \gamma]$. Let us prove that $g(x_\varepsilon(t)) \leq \varepsilon$ for all points $t \in [t_0, \gamma^*]$. Suppose by contradiction that $g(x_\varepsilon(\hat{t})) > \varepsilon$ for some $\hat{t} \in [t_0, \gamma^*]$. Take $\delta = (g(x_\varepsilon(\hat{t})) - \varepsilon)/2$ then $g(x_\varepsilon(\hat{t})) > \varepsilon + \delta$. Let

$$t^* = \min\{t : t \in [t_0, \gamma^*], g(x_\varepsilon(t)) = \varepsilon + \delta\}.$$

Then $g(x_\varepsilon(t^*)) = \varepsilon + \delta$ and due to the continuity of $g(x_\varepsilon(t))$ there exists $\beta > 0$ such that $g(x_\varepsilon(t)) > \varepsilon$ for $t^* - \beta \leq t \leq t^*$. For such t we have $h_\varepsilon(x_\varepsilon(t)) = 0$, hence

$$\frac{d}{dt} g(x_\varepsilon(t)) = (\nabla g(x_\varepsilon(t)), f(x_\varepsilon(t), \bar{u}(x_\varepsilon(t))) < 0.$$

The last inequality implies that $g(x_\varepsilon(t)) \geq \varepsilon + \delta$ for $t^* - \beta \leq t \leq t^*$, this contradicts the definition of t^*. Thus, $\gamma^* = \theta$ and for $t \in [t_0, \theta]$ the inequality $g(x_\varepsilon(t)) \leq \varepsilon$ holds.

To prove the final part of the theorem let us note that $g(x) \leq 0$ implies the equality $f_\varepsilon(x, u) = f(x, u)$ $\forall u \in U$, hence $G_0(\theta) \subset G_\varepsilon(\theta)$. By the NFT (neighboring feasible trajectory) theorem (see, e.g., [4, 5, 2]) assumption 2 implies the existence of a constant L with the following property. For any trajectory $x(t)$ of system (1) starting from the point $x(t_0) = x^0$ there exists a trajectory $\hat{x}(t)$, $\hat{x}(t_0) = x^0$, that satisfies the state constraints and the inequality

$$\max_{t_0 \leq t \leq \theta} \|x(t) - \hat{x}(t)\| \leq L \max_{t_0 \leq t \leq \theta} \max\{g(x(t)), 0\}.$$

Since the trajectory $x_\varepsilon(t)$ of the system (6) is a trajectory of the system (1), the last inequality implies that for every $x_\varepsilon(\theta) \in G_\varepsilon(\theta)$ there exists $\hat{x}(\theta) \in G_0(\theta)$ such that

$$\|x_\varepsilon(\theta) - \hat{x}(\theta)\| \leq L \max_{t_0 \leq t \leq \theta} \max\{g(x_\varepsilon(t)), 0\} \leq L\varepsilon.$$

Because $G_0(\theta) \subset G_\varepsilon(\theta)$, it implies the assertion of the theorem. □

Remark 1. The estimate (11) is uniform with respect to θ from a bounded set. If instead of reachable sets at the time θ, we consider the reachable sets until the time θ:

$$\bar{G}_0(\theta) = \bigcup_{0 \leq \tau \leq \theta} G_0(\tau), \quad \bar{G}_\varepsilon(\theta) = \bigcup_{0 \leq \tau \leq \theta} G_\varepsilon(\tau),$$

the estimate (11) remains valid for these sets.

The next theorem shows that condition 3 in assumption 1 may be omitted.

Theorem 2.2. *Let the function $f(x, u)$ and the constraints of the problem satisfy assumption 3 and conditions 1, 2 of assumption 1. Let $g(x^0) < 0$. Then the assertion of theorem 2.1 holds.*

Proof. Denote

$$P = \{\alpha \in \mathbb{R}^{(n+1)} : \alpha_i \geq 0, \ i = 1, \ldots, n+1, \ \sum_{i=1}^{n+1} \alpha_i = 1.\}$$

Let W be a set of vectors $w \in \mathbb{R}^{(n+1)(r+1)}$ such that $w = (u^1, \ldots, u^{(n+1)}, \alpha)$, $u^i \in U$, $\alpha \in P$. Thus $W = U \times \cdots \times U \times P$. Define a function $F(x, w) : \mathbb{R}^n \times W \to \mathbb{R}^n$ by the equality

$$F(x,w)=\sum_{i=1}^{n+1}\alpha_i f(x,u^i).$$

This function satisfies assumption 1, condition 3 of this assumption is valid because $F(x,W)=\mathrm{co}f(x,U)$.

By the hypothesis of the theorem there exist a $\sigma>0$ and a Lipschitz continuous feedback control $\bar{u}:S_R^\sigma\to U$ such that

$$(\nabla g(x),f(x,\bar{u}(x)))<0,\quad \forall x\in S_R^\sigma.$$

Setting $\bar{w}(x)=(\bar{u}(x),\hat{u},\dots,\hat{u},e^1)$ where $\hat{u}$ is any vector from U, $e^1=(1,0,\dots,0)^\top$ we get $f(x,\bar{u}(x))=F(x,\bar{w}(x))$, hence

$$(\nabla g(x),F(x,\bar{w}(x)))<0,\quad \forall x\in S_R^\sigma.$$

Let

$$F_\varepsilon(x,u)=\begin{cases} h_\varepsilon(g(x))F(x,u)+(1-h_\varepsilon(g(x)))F(x,\bar{w}(x)) & \text{if} \quad g(x)>0,\\ F(x,u) & \text{if} \quad g(x)\le 0,\end{cases}$$

then we have $F_\varepsilon(x,W)=\mathrm{co}f_\varepsilon(x,U)$.

Consider the control systems

$$\dot{x}(t)=F(x(t),w(t)),\ x(t_0)=x^0,\ w(t)\in W,\ t_0\le t\le\theta, \tag{12}$$

and

$$\dot{x}(t)=F_\varepsilon(x(t),w(t)),\ x(t_0)=x^0,\ w(t)\in W,\ t_0\le t\le\theta. \tag{13}$$

Denote by $\bar{G}_\varepsilon(\theta)$ the reachable set of system (13) without state constraints and by $\bar{G}_0(\theta)$ the reachable set of system (12) with state constraints $g(x(t))\le 0$, $t_0\le t\le\theta$. By theorem 2.1 there exists a positive number L not dependent on ε such that

$$h(\bar{G}_0(\theta),\bar{G}_\varepsilon(\theta))\le L\varepsilon.$$

The equality $F_\varepsilon(x,W)=\mathrm{co}f_\varepsilon(x,U)$ implies that any trajectory of system (13) may be approximated by the trajectories of system (1) arbitrarily closely in the uniform metric [10]. From this it follows that

$$\bar{G}_\varepsilon(\theta)=\mathrm{cl}\bar{G}_0(\theta).$$

Let $\bar{x}(t)$ be any trajectory of system (12) satisfying the state constraints. By [19, Lemma 2.2], $\bar{x}(t)$ may be approximated arbitrarily closely by the trajectory $\hat{x}(t)$ of this system such that $\bar{x}(t)\in\mathrm{int}S$, that is $g(\bar{x}(t))<0,\ t_0\le t\le\theta$. In turn, $\hat{x}(t)$ admits an approximation by the trajectory $x(t)$ of system (1) such that $g(x(t))<0,\ t_0\le t\le\theta$. This implies equality

$$\bar{G}_0(\theta)=\mathrm{cl}G_0(\theta).$$

In view of the equality $h(A,B)=h(\mathrm{cl}A,\mathrm{cl}B)$ for any pair of nonempty subsets of $\mathbb{R}^n$ we get

$$h(G_0(\theta),G_\varepsilon(\theta))\le L\varepsilon.$$

□

3. **Control-affine nonlinear system with ellipsoid constraints on control.** Consider the control system

$$\dot{x}(t) = f(x(t), u(t)) = f_1(x(t)) + f_2(x(t))u(t),\ u(t) \in U,\ x(t_0) = x^0,$$

where $f_1 : \mathbb{R}^n \to \mathbb{R}^n$, $f_2 : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are continuously differentiable mappings, $\mathbb{R}^{n\times r}$ is a linear space of $n \times r$ real matrices. Here, we assume that the values of the control u belongs to a non-degenerate ellipsoid in $\mathbb{R}^r$:

$$U = \{u \in \mathbb{R}^r : (u - \hat{u})^\top Q(u - \hat{u}) \leq 1\},$$

where Q is a positive definite $r \times r$ matrix, $\hat{u} \in \mathbb{R}^r$ is a center of the ellipsoid. Assume that the function $\nabla g(x)$ is Lipschitz continuous.

Assumption 2 for the system may be rewritten in the form

$$\nabla g(x)^\top f_1(x) + \nabla g(x)^\top f_2(x)\hat{u} + \min_{v\in V} \nabla g(x)^\top f_2(x)v < 0, \tag{14}$$

for $x \in S_R^\sigma$. Here $V = \{v : v^\top Qv \leq 1\}$ is an ellipsoid centered at the origin..

Introduce the following notation

$$a(x) = \nabla g(x)^\top f_1(x) + \nabla g(x)^\top f_2(x)\hat{u}, \quad b^\top(x) = \nabla g(x)^\top f_2(x),$$

here $a(x)$ is a scalar and $b(x)$ is an r-vector. We have

$$\min_{v\in V} b^\top(x)v = \min_{(w,w)\leq 1} b^\top(x)Q^{-\frac{1}{2}}w = -\|Q^{-\frac{1}{2}}b(x)\| = -\sqrt{b^\top(x)Q^{-1}b(x)},$$

where $Q^{-\frac{1}{2}} = (Q^{-1})^{\frac{1}{2}}$ is the square root of a positive definite matrix Q^{-1}. Given this notation condition (14) takes the form

$$a(x) + \min_{v\in V} b^\top(x)v = a(x) - \sqrt{b^\top(x)Q^{-1}b(x)} < 0. \tag{15}$$

The minimum in (14) is attained on a vector $v = v(x)$, where

$$v(x) = \frac{Q^{-1}b(x)}{\sqrt{b^\top(x)Q^{-1}b(x)}}. \tag{16}$$

Thus, for the function $\bar{u}(x) = v(x) + \hat{u}$ the inequality $(\nabla g(x), f(x, \bar{u}(x))) < 0$ holds. However, generally speaking, this function is not Lipschitz continuous, it may even be discontinuous at the points x, where $b(x) = 0$. We show that by modifying formula (16) we can ensure Lipschitz continuity of the function $\bar{u}(x)$.

Theorem 3.1. *Let the condition (15) which is equivalent to (14) be fulfilled on the set S_R^σ. Then there exists a Lipschitz function $\bar{u}(x)$ such that*

$$(\nabla g(x), f_1(x) + f_2(x)\bar{u}(x))) < 0,\ \forall x \in S_R^\sigma. \tag{17}$$

Proof. Inequality (14) implies that $a(x) < 0$ at the points x, where $b(x) = 0$. We substitute $w = Q^{\frac{1}{2}}v$, then $a(x) + b^\top(x)v = a(x) + b_1^\top(x)w$, where $b_1(x) = Q^{-\frac{1}{2}}b(x)$. The ellipsoid V transforms to the ball $\{w : (w, w) \leq 1\}$. Consider a non-negative function $p(x)$ defined on S_R^σ by the equality

$$p(x) = \begin{cases} \frac{a(x)+\sqrt{a^2(x)+(b_1^\top(x)b_1(x))^2}}{b_1^\top(x)b_1(x)} & \text{if} \quad b_1(x) \neq 0, \\ 0 & \text{if} \quad b_1(x) = 0. \end{cases}$$

Since $a(x) < 0$ for $b_1(x) = 0$, the function $p(x)$ is continuously differentiable [18]. Sontag feedback control [1] $\bar{w}(x) = -p(x)b_1(x)$ satisfies the inequality $a(x) +$

[1]the formula for $\bar{w}$ coincides with the formula for the Sontag stabilizing control if in the latter we replace the Lyapunov function by $g(x)$

$b_1^\top(x)\bar{w}(x) < 0$, hence for $\bar{u}(x) = Q^{-\frac{1}{2}}\bar{w}(x) + \hat{u}$ the inequality $(\nabla g(x), f(x, \bar{u}(x))) < 0$ holds. The function $\bar{u}(x)$ is obviously Lipschitz. To ensure the condition $\bar{u}(x) \in U$ let us modify $\bar{w}(x)$ as follows. Let $\pi(w)$ be the operator of metric projection onto the unit Euclidean ball in $\mathbb{R}^r$, $\pi(w)$ satisfies the Lipschitz condition with the Lipschitz constant equal to unit. Set

$$\bar{w}(x) = \pi(-p(x)b_1(x)) = \begin{cases} -p(x)b_1(x), & \text{if} \quad \|p(x)b_1(x)\| \le 1, \\ \frac{-p(x)b_1(x)}{\|p(x)b_1(x)\|} = -\frac{b_1(x)}{\|b_1(x)\|}, & \text{if} \quad \|p(x)b_1(x)\| > 1. \end{cases}$$

If $\|p(x)b_1(x)\| \le 1$ then $\bar{w}(x) = -p(x)b_1(x)$, hence $a(x) + b_1^\top(x)\bar{w}(x) < 0$. If $\|p(x)b_1(x)\| > 1$ we have

$$a(x) + b_1^\top(x)\bar{w}(x) = a(x) - \|b_1(x)\| = a(x) - \sqrt{b^\top(x)Q^{-1}b(x)} < 0$$

due to (15). Thus

$$\bar{u}(x) = Q^{-\frac{1}{2}}\pi(-p(x)Q^{-\frac{1}{2}}f_2^\top(x)\nabla g(x)) + \hat{u}$$

is the required control function. The theorem is proved. □

Remark 2. From the definition of the auxiliary system and the proof of theorems 2.1, 2.2, 3.1 it is clear that the continuous differentiability of the function g on the set $\{x \in \mathbb{R}^n : g(x) < 0\}$ need not be assumed. It is sufficient to consider g to be continuous on $\mathbb{R}^n$ and to be continuously differentiable on the set $\{x \in \mathbb{R}^n : 0 \le g(x) < \sigma\}$ for some $\sigma > 0$.

4. Examples.

Consider the control system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad x_1(0) = 0,\ x_2(0) = 0,\ |u| \le 1,\ 0 \le t \le 2, \tag{18}$$

with the set of state constrains $S = \{x \in \mathbb{R}^2 : |x_2| - 1 \le 0\}$. The function $g(x) = |x_2| - 1$ satisfies the conditions of Remark 2. Hereinafter $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$.

Let us take the function $h_\varepsilon(\tau)$ in the form (9). There exists a control $\bar{u}(x)$ such that $g'(x_2)\bar{u}(x) < 0$ in the neighborhood of $\partial S = \{x \in \mathbb{R}^2 : |x_2| = 1\}$, e.g.,

$$\bar{u}(x) = \begin{cases} -1 & \text{if} \quad x_2 \ge 1, \\ -x_2 & \text{if} \quad -1 < x_2 < 1, \\ 1 & \text{if} \quad x_2 \le -1. \end{cases}$$

The auxiliary system takes the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = p_\varepsilon(x_2, u), \tag{19}$$

where $p_\varepsilon(x_2, u) = h_\varepsilon(|x_2| - 1)\, u + (1 - h_\varepsilon(|x_2| - 1))\, \bar{u}(x)$. Note that the first equation of the system (18) remains unchanged as this equation does not contain u.

It is known (see, e.g., [15, Theorem 3 on p. 254]) that any control that steers the trajectory of the system to the boundary of the reachable set, satisfies the Pontryagin maximum principle. The Hamiltonian of the system (19) has the form $H(x, \psi) = \psi_1 x_2 + \psi_2\, p_\varepsilon(x_2, u)$. From the maximum principle, we have $u(t) = \operatorname{sign}\psi_2(t)$, $t \in [0, 2]$, where $\psi(t)$ is the solution of the adjoint system

$$\begin{aligned} \dot{\psi}_1 &= -\frac{\partial H}{\partial x_1} = 0 \\ \dot{\psi}_2 &= -\frac{\partial H}{\partial x_2} = -\psi_1 - \psi_2\, p'_\varepsilon(x_2, u), \end{aligned}$$

such that $\psi_1^2(0) + \psi_2^2(0) = 1$, $p'_\varepsilon(x_2, u)$ is the derivative of p_ε in x_2.

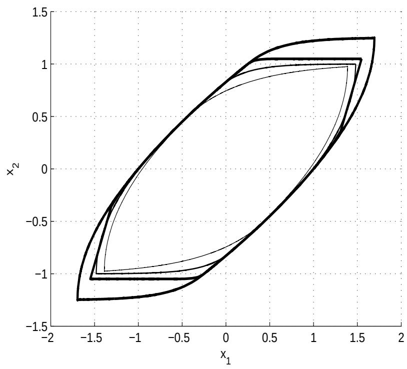

Figure 1: Internal and external approximating reachable sets of system (18)

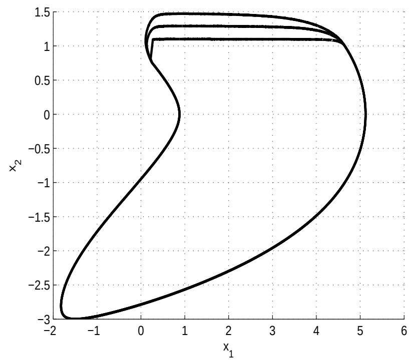

Figure 2: External approximating reachable sets of system (21) for different ε

Thus, any trajectory $x(t)$ of the system (19) with $x(2)$ on the boundary of the reachable set $G_\varepsilon(2)$ is contained among solutions of the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= p_\varepsilon(x_2, \operatorname{sign}\psi_2), \\ \dot{\psi}_1 &= 0 \\ \dot{\psi}_2 &= -\psi_1 - \psi_2\, p'_\varepsilon(x_2, \operatorname{sign}\psi_2). \end{aligned} \tag{20}$$

with initial states $x_1(0) = 0,\ x_2(0) = 0,\ \psi_1^2(0) + \psi_2^2(0) = 1$. Any of these initial states may be obtained by setting $\psi_1(0) = \sin\alpha,\ \psi_2(0) = \cos\alpha$, where $\alpha \in [0, 2\pi]$. Integrating system (20) for all $\alpha \in [0, 2\pi]$ we get a family of points $\begin{pmatrix} x_1(2,\alpha) \\ x_2(2,\alpha) \end{pmatrix}$ in the plane containing all the boundary points of the reachable set $G_\varepsilon(2)$ of system (19). The figure 1 shows the results of numerical simulation for the given algorithm. Here, the boundaries of approximating sets are drawn. Thick line shows the boundary of $G_\varepsilon(2)$ for different values of ε. For comparison, the thin line indicates the boundaries of internal approximating sets obtained by the method described in the paper [9].

Consider another example. Let the control system be

$$\dot{x}_1 = 1 - p\, x_2^2 + u_1, \quad \dot{x}_2 = u_2, \quad x_1(0) = 0,\ x_2(0) = 0, \quad 0 \le t \le 3, \tag{21}$$

where $p > 0$ and the constraints are given as follows : $u_1^2 + u_2^2 \le 1, \quad x_2 \le 1$. As a control $\bar{u}(x)$ one can take $\bar{u}(x) = \begin{pmatrix} \bar{u}_1(x) \\ \bar{u}_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\forall x \in \mathbb{R}^2$. Then the auxiliary system takes the form

$$\dot{x}_1 = 1 - p\, x_2^2 + h_\varepsilon(x_2 - 1)u_1, \quad \dot{x}_2 = h_\varepsilon(x_2 - 1)(1 + u_2) - 1.$$

The algorithm for constructing the boundary of reachable sets is similar to the first example. The figure 2 shows the result of constructing boundaries of reachable sets $G_\varepsilon(3)$ for $p = 0.5$ and for different values of ε. The boundaries of these sets in the bottom of the figure coincide.

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E-mail address: gmi@imm.uran.ru