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Factorization of the Reaction–Diffusion Equation, the Wave Equation, and Other Equations

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Abstract—We investigate equations of the form $D_t u = \Delta u + \xi\nabla u$ for an unknown function $u(t,x)$, $t \in \mathbb{R}$, $x \in X$, where $D_t u = a_0(u,t) + \sum_{k=1}^{r} a_k(t,u)\partial_t^k u$, Δ is the Laplace−Beltrami operator on a Riemannian manifold X, and ξ is a smooth vector field on X. More exactly, we study morphisms from this equation within the category $\mathcal{PDE}$ of partial differential equations, which was introduced by the author earlier. We restrict ourselves to morphisms of a special form — the so-called *geometric morphisms*, which are given by maps of X to other smooth manifolds (of the same or smaller dimension).

It is shown that a map $f\colon X \to Y$ defines a morphism from the equation $D_t u = \Delta u + \xi\nabla u$ if and only if, for some vector field Ξ and a metric on Y, the equality $(\Delta+\xi\nabla)f^*v = f^*(\Delta+\Xi\nabla)v$ holds for any smooth function $v\colon Y \to \mathbb{R}$. In this case, the quotient equation is $D_t v = \Delta v + \Xi\nabla v$ for the unknown function $v(t,y)$, $y \in Y$.

It is also shown that, if a map $f\colon X \to Y$ is a locally trivial bundle, then f defines a morphism from the equation $D_t u = \Delta u$ if and only if fibers of f are parallel and, for any path γ on Y, the expansion factor of a fiber transferred along the horizontal lift γ to X depends on γ only.
Keywords: category of partial differential equations, reaction–diffusion equation, heat equation, wave equation.

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INTRODUCTION

In [1] the author defined the category $\mathcal{PDE}$ of partial differential equations. In Section 1 we give a strict definition of this category; in simple words, its objects are partial differential equations, and morphisms from an equation E to an equation E' are maps from the space N of dependent and independent variables of E to the space N' of dependent and independent variables of E' such that a submanifold $\Gamma \subset N'$ is the graph of a solution of E' if and only if its preimage $F^{-1}(\Gamma)$ is the graph of a solution of E. Such an equation E' will be called a *factor-equation* for E.

A special case of a morphism from E is the factorization by the symmetry group of the equation E. More exactly, if G is a group of transformations of N that fixes the equation E and the factor-map $N \to N/G$ is a locally trivial bundle, then this map defines a morphism from E to the equation E/G, which describes a solution of E invariant under G. Symmetry groups of differential equations are widely used for constructing particular solutions, which are known as

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invariant and partially invariant solutions. However, the stock of morphisms in the category $\mathcal{PDE}$ is generally essentially larger than the stock of factorizations by symmetry groups [1, 2], which makes it possible to use these morphisms for finding new classes of solutions to partial differential equations or for performing a qualitative study of such classes. In addition, in the investigation of the internal structure of specific subcategories of $\mathcal{PDE}$, a natural classification of equations of this subcategory appears (see, e.g., the classification of second-order parabolic equations in [1]).

In this paper, we study morphisms of the category $\mathcal{PDE}$ from equations of the form

$$D_t u = \Delta u + L_\xi u \tag{0.1}$$

as well as from the narrower class of equations of the form

$$D_t u = \Delta u, \tag{0.2}$$

where D_t is the differential operator

$$D_t u = a_0(t,u) + \sum_{k=1}^{r} a_k(t,u)\partial_t^k u,$$

$r \geq 0$ (for $r = 0$, we assume that $D_t u = a_0(t,u)$), and $a_k(t,u)$ are continuous real-valued functions such that, for any pair (t,u), at least one of the coefficients $a_k(t,u)$ is nonzero. Here, Δ is the Laplace–Beltrami operator on a Riemannian manifold X: $\Delta u = \operatorname{div}(\nabla u)$, ξ is a smooth vector field on X, and $L_\xi u$ is the derivative of the function u along ξ.

Special cases of equation (0.1) are the reaction–diffusion equation $u_t = a(u)(\Delta u + L_\xi u) + b(u)$, the nonlinear heat equation $u_t = a(u)\Delta u$, the wave equation $u_{tt} = a(u)\Delta u$, the elliptic equation $\Delta u = b(u)$, and many others.

In general, morphisms from equation (0.1) are given by maps $F\colon N \to N'$ (where $N = \mathbb{R} \times X \times \mathbb{R}$ and N' is an arbitrary smooth manifold) satisfying additional conditions (the definition of a morphism). However, in this paper we restrict ourselves to considering a narrower class of maps, more exactly, morphisms of the form

$$F = (\mathrm{id}, f, \mathrm{id})\colon (t,x,u) \mapsto (t, f(x), u) \tag{0.3}$$

given by a smooth map $f\colon X \to Y$, where Y is an arbitrary smooth manifold and $N' = \mathbb{R} \times Y \times \mathbb{R}$. In this paper, morphisms of the category $\mathcal{PDE}$ from equations (0.1) and (0.2) satisfying condition (0.3) will be called for brevity *geometric morphisms*. Note that, by the definition of a morphism in the category $\mathcal{PDE}$, f must be a surjective submersion.

It will be shown below that, in particular, for any geometric morphism from equation (0.1), the corresponding factor-equation (for an unknown function $v\colon \mathbb{R} \times Y \to \mathbb{R}$) has the form

$$D_t v = \Delta_Y v + L_\Xi v \tag{0.4}$$

with the same operator D_t, where Δ_Y is the Laplace–Beltrami operator on a Riemannian manifold Y and Ξ is a smooth vector field on Y. By the definition of a morphism in the category $\mathcal{PDE}$, any solution $v(t,y)$ of equation (0.4) corresponds to a solution $u(t,x) = v(t, f(x))$ of equation (0.1) and, vice versa, if a solution of equation (0.1) can be represented in the form $u(t,x) = v(t, f(x))$ for some function $v(t,y)$, then $v(t,y)$ is a solution of equation (0.4).

The general description of geometric morphisms from equation (0.1) obtained in this paper makes it possible to perform their further study in more detail, which will be done in a forthcoming paper.

To justify constraint (0.3), we can say the following. On the one hand, this class of morphisms is sufficiently large, and its consideration is sufficient in many situations. In particular, it can be used to show the relation between morphisms in $\mathcal{PDE}$ and factorizations of the original equation by symmetry groups. On the other hand, it is shown in [3] that, for the coefficient $a(u)$ of a rather general form, any morphism from the equation $u_t = a(u)(\Delta u + L_\xi u) + b(u)$ within the category $\mathcal{PE}$ of parabolic equations defined there can be reduced to form (0.3) by a bijective change of variables in the factor-equation.

1. THE CATEGORY $\mathcal{PDE}$

Let us first recall some definitions. Suppose that N is a smooth manifold, d and s are positive integers, and $s < \dim N$. The *bundle of d-jets* (submanifolds of codimension s) $\pi^d\colon J^d_s(N) \to N$ is the bundle with fiber $J^d_s(N)\big|_x$ over a point $x \in N$, where $J^d_s(N)\big|_x$ is the set of equivalence classes of smooth submanifolds L with codimension s in N passing through x under the equivalence relation of the dth-order contact at the point x. By definition, a *d-jet* of a smooth submanifold $L \subset N$ at a point $x \in L$ is the equivalence class from $J^d_s(N)\big|_x$ with representative L. The *prolongation* map $j^d_L\colon L \to J^d_s(N)$ takes a point $x \in L$ to the d-jet of L at x. A *differential equation of order d* will be understood as an arbitrary subset E of the jet space $J^d_s(N)$. A smooth submanifold L of codimension s in N is called a *solution* of E if the image $j^d_L(L)$ of its dth prolongation is contained in E.

The definition of a differential equation given here is very general; most applications involve equations of "classical" form, where E is a closed subset (or even a submanifold) of the space $J^d(M;S)$ of d-jets of smooth maps from M to S. If $E \subset J^d(M;S)$ is such a classical differential equation, then its *extended version* is the closure of E in $J^d_s(M \times S)$, where $s = \dim S$ [4]. Here, $J^d(M;S)$ is identified with the open subset in $J^d_s(M \times S)$ consisting of d-jets of graphs of all possible smooth maps from M to S. Obviously, a smooth map $u\colon M \to S$ is a solution of a classical equation E if and only if its graph is a solution of the extended version of E. However, the extended version of E also admits multivalued solutions and solutions with infinite derivatives (see details in [4]). Further, when we speak about morphisms from a specific equation of classical form (for example, from equation (0.1) or (0.2)), this will always imply morphisms from its extended version, i.e., from the closure of the corresponding set $E \subset J^d(M;S)$ to $J^d_s(M \times S)$.

Now we can formulate the definition of the category $\mathcal{PDE}$ given in [1].

For an arbitrary map $F\colon N \to N'$, a subset $L \subset N$ will be called *F-projected* if $L = F^{-1}(F(L))$.

Let N and N' be smooth submanifolds, and let $F\colon N \to N'$ be a surjective submersion. The *F-projected jet bundle* $J^d_{s,F}(N)$ is the submanifold $J^d_s(N)$ consisting of d-jets all possible F-projected submanifolds N of codimension s with induced structure of the bundle over N.

There is a natural isomorphism between the bundles $J^d_{s,F}(N)$ and $F^*J^d_s(N')$, where $F^*J^d_s(N') = J^d_s(N') \times_{N'} N$ is the pullback of $J^d_s(N')$ along the map F. Therefore, we can lift F to the map $F^d\colon J^d_{s,F}(N) \to J^d_s(N')$, which is an isomorphism on fibers, in the following natural way (Fig. 1). Let $\vartheta \in J^d_{s,F}(N)$.

1. Take an arbitrary F-projected manifold $L \subset N$ such that its dth prolongation L passes through ϑ (in other words, the d-jet of L at the point $\pi^d(\vartheta)$ is ϑ).

2. Assign to ϑ the point $\vartheta' \in J^d_s(N')$ that is the d-jet of the submanifold $L' = F(L) \subset N'$ at the point $F \circ \pi^d(\vartheta)$. (For an F-projected smooth submanifold $L \subset N$, its image $F(L)$ is a smooth submanifold of N'.)

$$\begin{array}{ccc} J^d_{s,F}(N) & \xrightarrow{F^d} & J^d_s(N') \\ \downarrow{\scriptstyle \pi^d} & & \downarrow{\scriptstyle \pi'^d} \\ N & \xrightarrow{F} & N' \end{array}$$

Fig. 1.

$$\begin{array}{ccccc} E & \hookleftarrow & E \cap J^d_{s,F}(N) & \twoheadrightarrow & E' \\ \downarrow & & \downarrow & & \downarrow \\ J^d_s(N) & \hookleftarrow & J^d_{s,F}(N) & \xrightarrow{F^d} & J^d_s(N') \\ \downarrow{\scriptstyle \pi^d} & & \downarrow & & \downarrow{\scriptstyle \pi'^d} \\ N & = & N & \xrightarrow{F} & N' \end{array}$$

Fig. 2. Admissible maps.

Definition 1 [1, Definition 1]. Let $F\colon N \to N'$ be a smooth surjective submersion. A subset $E \subset J^d_s(N)$ *admits* F if $E \cap J^d_{s,F}(N)$ is an F^d-projected subset of $J^d_{s,F}(N)$ (Fig. 2). Equivalently, $E \cap J^d_{s,F}(N)$ is the preimage $\left(F^d\right)^{-1}(E')$ of some subset $E' \subset J^d_s(N')$; this subset E' is called the *F-projection* of E.

Definition 2 [1, Definition 2]. The *category of partial differential equations* $\mathcal{PDE}$ is defined as follows: its objects are pairs (N, E), where N is a smooth manifold and E is a subset of $J^d_s(N)$ for some natural $d, s \geq 1$, and morphisms from (N, E) to (N', E') are surjective submersions $F\colon N \to N'$ admitted by E such that E' is the F-projection of E. The composition of morphisms is defined as the composition of the corresponding maps.

Note that the category $\mathcal{PDE}$ defined here differs essentially from the category of nonlinear differential equations DE defined in [5, 6], which should be taken into account to avoid confusion. The general definition of DE from [5,6] can be formulated in a simplified form as follows [7]: objects of DE are infinite-dimensional manifolds equipped with a completely integrable finite-dimensional distribution (in particular, infinitely prolongated differential equations), and its morphisms are smooth maps for which the image of the distribution is contained in the distribution on the image. In simpler terms, a factor-object in this category is an equation in the factor-space that describes images (under projection to the factor-space) of arbitrary solutions of the original equation; in this approach, from each factor-object we obtain some information about all solutions of the original equation. In contrast, a factor-object in the category $\mathcal{PDE}$ is an equation such that preimages of all its solutions are solutions of the original equation; here, from each factor-object we obtain the complete information about some class of solutions of the original equation.

2. GEOMETRIC MORPHISMS

Theorem 1. *Suppose that X is a smooth Riemannian manifold, Y is a smooth manifold, $f\colon X \to Y$ is a surjective submersion, ξ is a smooth vector field on X, and $a_k(t, u)$ are continuous functions such that, for any pair (t, u), at least one of the coefficients $a_k(t, u)$ is nonzero. Then the following two conditions are equivalent*:

1. *Map* (0.3) *is a morphism of the category $\mathcal{PDE}$ from equation* (0.1).

2. *There exist a vector field Ξ and a Riemannian metric on Y such that the diagram in Fig.* 3 *is commutative.*

In this case, the factor-equation (*for the unknown function $v\colon \mathbb{R} \times Y \to \mathbb{R}$*) *has form* (0.4) *with the same operator D_t*:

$$D_t v = \Delta v + L_\Xi v.$$

$$\begin{array}{ccc} C^\infty(Y) & \xrightarrow{v\mapsto \Delta v+L_\Xi v} & C^\infty(Y)\\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle f^*}\\ C^\infty(X) & \xrightarrow{u\mapsto \Delta u+L_\xi u} & C^\infty(X)\end{array}$$

Fig. 3.

$$\begin{array}{ccccc} E_F & \hookrightarrow & J_F(M;S) & \longrightarrow & J(M';S)\\ \big\downarrow & & \big\downarrow & & \big\downarrow\\ \overline{E}_F & \hookrightarrow & J_F(N) & \xrightarrow{F^d} & J(N')\\ & & \big\downarrow{\scriptstyle \pi^d} & & \big\downarrow{\scriptstyle \pi'^d}\\ & & N & \xrightarrow{F} & N' \end{array} \qquad \begin{array}{ccccc} E_{F,y} & \hookrightarrow & N_y\times U_y & \longrightarrow & U_y\\ \big\downarrow & & \big\downarrow & & \big\downarrow\\ \overline{E}_{F,y} & \hookrightarrow & N_y\times Z_y & \longrightarrow & Z_y\\ & & \big\downarrow & & \big\downarrow\\ & & N_y & \longrightarrow & \{y\} \end{array}$$

Fig. 4. Base replacement.

Here, the same letter $\Delta = \Delta_Y$ denotes the Laplace–Beltrami operator on the Riemannian manifold Y.

Remark. If Y is simply connected, then the differential form dual to the vector field Ξ on the Riemannian manifold Y is exact and factor-equation (0.4) can be written in the form

$$D_t v = \varphi^{-1}\operatorname{div}(\varphi\nabla v)$$

for some smooth function $\varphi\colon Y\to\mathbb{R}_+$ (see also Corollary 2).

In the proof of the theorem we will need the following general fact. Suppose that M, M', and S are smooth manifolds, $s=\dim S$, $N=M\times S$, $N'=M'\times S$, $\varphi\colon M\to M'$ is a smooth surjective submersion, and $F=\varphi\times\mathrm{id}\colon N\to N'$. Recall that the space $J^d(M;S)$ of d-jets for maps from M to S is identified with the open subset in $J^d_s(N)$ consisting of d-jets of graphs of all possible smooth maps from M to S. Denote by $J^d_F(M;S)=J^d(M;S)\cap J^d_{s,F}(N)$ the set of d-jets of F-projected graphs for maps from M to S.

Lemma 1. *Let E be a closed subset of $J^d(M;S)$, and let $\overline{E}$ be the closure of E in $J^d_s(N)$. Then the following two conditions are equivalent*:

1. *$E\cap J^d_F(M;S)$ is an F^d-projected subset of $J^d_F(M;S)$.*
2. *$\overline{E}\cap J^d_{s,F}(N)$ is an F^d-projected subset of $J^d_{s,F}(N)$.*

Proof. To simplify formulas within this proof, we will write briefly J instead of J^d_s and J^d and also J_F instead of $J^d_{s,F}$ and J^d_F.

Note first that $J_F(N)\cong F^*J(N')$ and $J_F(M;S)\cong F^*J(M';S)$ as bundles over N.

Define $E_F=E\cap J_F(M;S)=E\cap J_F(N)$ and $\overline{E}_F=\overline{E}\cap J_F(N)$. The subset $J_F(N)$ is closed in $J(N)$; hence, $\overline{E}_F$ coincides with the closure of E_F in $J_F(N)$.

Denote by Z_y the fiber of $J(N')$ over a point $y\in N'$ and by U_y the fiber of $J(M';S)$ over y. If we replace the base on the left diagram in Fig. 4 by means of the inclusion $\{y\}\hookrightarrow N'$, then we obtain the right diagram Fig. 4, where $N_y=F^{-1}(y)$, $\overline{E}_{F,y}=\overline{E}_F\cap(F^d)^{-1}(Z_y)$, and $E_{F,y}=E_F\cap(F^d)^{-1}(U_y)$.

Since Z_y is closed in $J(N')$, its preimage $(F^d)^{-1}(Z_y)$ is closed in $(F^d)^{-1}(J(N')) = J_F(N)$ and $\overline{E}_{F,y}$ coincides with the closure of $E_{F,y}$ in $(F^d)^{-1}(Z_y) \cong N_y \times Z_y$. Thus, it is sufficient to prove that, for any point $y \in N'$, the following two conditions are equivalent:

(1) $E_{F,y}$ has the form $N_y \times A$ for some subset $A \subset U_y$;

(2) the closure of $E_{F,y}$ in $N_y \times Z_y$ has the form $N_y \times B$ for some subset $B \subset Z_y$.

Assume that $E_{F,y} = N_y \times A$; then, obviously, $\overline{E}_{F,y} = N_y \times B$, where B is the closure of A in Z_y. Let, vice versa, $\overline{E}_{F,y} = N_y \times B$. Then $E_{F,y} = (N_y \times U_y) \cap \overline{E}_{F,y} = N_y \times (B \cap U_y)$, since $E_{F,y}$ is closed in $N_y \times U_y$ (because E is closed in $J(M;S)$ by the hypothesis of the lemma). Thus, conditions (1) and (2) are equivalent, which completes the proof of the lemma. □

Proof of Theorem 1. In our situation, $M = \mathbb{R} \times X$, $M' = \mathbb{R} \times Y$, $\varphi = \mathrm{id} \times f$, $S = \mathbb{R}$, $d = \max(2, r)$, and equation (0.1) defines a closed subset E in $J^d(M;\mathbb{R})$. By definition, map (0.3) specifies a morphism of the category $\mathcal{PDE}$ from the "classical" equation E if $\overline{E} \cap J^d_{s,F}(N)$ is an F^d-projected subset of $J^d_{s,F}(N)$. By Lemma 1, this condition is equivalent to the fact that $E \cap J^d_F(M;S)$ is an F^d-projected subset of $J^d_F(M;S)$.

Implication 1 ⇒ 2. Assume that ϑ is an arbitrary point from $E \cap J^d_F(M;\mathbb{R})$, $\pi_d(\vartheta) = (t,x,u)$, and $f(x) = y$. We take a function $\tilde{v}\colon M' \to \mathbb{R}$ whose d-jet at the point (t,y) coincides with $F^d(\vartheta) \in J^d(M';\mathbb{R})$; then, the d-jet of the function $\tilde{u} = \varphi^*\tilde{v}$ at the point (t,x) coincides with ϑ. Since ϑ lies in E, the value of the expression

$$D_t\tilde{u} - \Delta\tilde{u} - L_\xi\tilde{u} \tag{2.1}$$

at the point (t,x) is zero.

Let V be the chart of Y containing the point y, and let (y^i) be local coordinates on V. The map f is written in these coordinates as $f(x) = \big(f^1(x),\dots,f^m(x)\big)$ for $x \in f^{-1}(V)$, where $m = \dim Y$. Expressing value (2.1) at the point (t,x) in terms of the d-jet $\tilde{v}$ at the point (t,y), we obtain in the coordinate notation

$$a_0(t,u) + \sum_{k=1}^{r} a_k(t,u) v_{\underbrace{t..t}_{k}} - \sum_{i,j=1}^{m} g^{ij}(x) v_{ij} - \sum_{i=1}^{m} \zeta^i(x) v_i = 0, \tag{2.2}$$

where

$$v_i = \partial_i\tilde{v}(t,y), \quad v_{ij} = \partial_i\partial_j\tilde{v}(t,y), \quad v_{\underbrace{t..t}_{k}} = \partial_t^k\tilde{v}(t,y)$$

denote components of the d-jet of the function $\tilde{v}$ at the point (t,y), ∂_i is the partial derivative with respect to y^i, and the functions $g^{ij}, \zeta^i\colon f^{-1}(V) \to \mathbb{R}$ are defined by the formulas $g^{ij} = \big\langle \mathrm{d}f^i, \mathrm{d}f^j \big\rangle$ and $\zeta^i = \Delta f^i + L_\xi f^i$.

For fixed t, x, and u, equation (2.2) defines a subset $Q(t,x,u)$ of the fiber $J^d(\mathbb{R} \times V;\mathbb{R})$ over (t,y,u). By definition, the F^d-projectedness of $E \cap J^d_F(\mathbb{R} \times f^{-1}(V);\mathbb{R})$ is equivalent to the fact that, for all $y \in V$ and $t,u \in \mathbb{R}$, the set $Q(t,x,u) = E' \cap J^d(\mathbb{R} \times V;\mathbb{R})_{(t,y,u)}$ is independent of the choice of the point $x \in f^{-1}(y)$. Since, by the hypothesis of the theorem, at least one of the values $a_k(t,u)$ is not zero, this condition is equivalent to the f-projectedness from $f^{-1}(V)$ to V of the functions g^{ij} and ζ^i for all $i,j = 1\dots m$. In other words,

$$\begin{cases} \big\langle \mathrm{d}f^i, \mathrm{d}f^j \big\rangle_x = g^{ij}(f(x)), \\ \big(\Delta f^i + L_\xi f^i\big)_x = \zeta^i(f(x)) \end{cases} \tag{2.3}$$

for some functions g^{ij} and ζ^i on V (for convenience, we leave the same notation for these new functions). Since the form g^{ij} is positive definite, it specifies a Riemannian metric on V. Therefore, we find from (2.3) that

$$\Delta(f^*w) + L_\xi(f^*w) = f^*\left(\sum_{i,j=1}^{m} g^{ij}w_{ij} + \sum_{i=1}^{m}\zeta^i w_i\right) = f^*\left(\Delta w + L_\Xi w\right) \tag{2.4}$$

for an arbitrary function $w\colon V \to \mathbb{R}$, where the Laplace−Beltrami operator on V is taken with respect to the Riemannian metric $g = (g^{ij})$ and the vector field Ξ on V is defined by the formula

$$\Xi^i = \zeta^i - \frac{1}{\sqrt{|g|}}\partial_j\left(\sqrt{|g|}g^{ij}\right), \quad |g| = |\det(g_{ij})|.$$

Choose a cover of Y by charts diffeomorphic to $\mathbb{R}^m$. Under a coordinate change, $\Xi^i(y)$ and $g^{ij}(y)$ behave as sections of TY and of the symmetric square of TY, respectively. Hence, after gluing the charts, they specify globally a vector field Ξ and a Riemannian metric g on Y. By (2.4), the identity $\Delta(f^*w) + L_\xi(f^*w) = f^*\left(\Delta w + L_\Xi w\right)$ holds for an arbitrary function $w\colon Y \to \mathbb{R}$ on each chart, which means that it holds globally on Y. This proves the implication $(1 \Rightarrow 2)$ in the statement of Theorem 1.

The implication $(2 \Rightarrow 1)$ is proved by repeating the argument in the reverse order. □

3. FACTORIZATION WITH DIMENSION REDUCTION

Assume that X is a smooth Riemannian manifold, Y is a smooth manifold, and $f\colon X \to Y$ is a surjective submersion. Let $n = \dim X$, $m = \dim Y$, and $k = n - m$. In general, $n \geq m$; from now on, we will consider the case $n > m$. Then, the fibers of f (preimages $f^{-1}(y)$, $y \in Y$) are smooth submanifolds of X of dimension $k > 0$ [8].

A 1-form on X will be called horizontal if it has zero restriction to any fiber. Equivalently, a 1-form is called horizontal if it lies in a subbundle f^*T^*Y of the bundle T^*X. Similarly, an s-form on X is called horizontal if it lies in a subbundle $f^*\Lambda^sT^*Y$ of the bundle Λ^sT^*X, where Λ^sT^*X is the bundle of differential s-forms over X.

The tangent bundle TX is split into the direct sum $TX = T_vX + T_hX$, where the vertical bundle $T_vX = \{\eta \in TX\colon f_*\eta = 0\}$ consists of vectors tangent to fibers and the horizontal bundle T_hX consists of vectors orthogonal to fibers. For an arbitrary vector field η on Y, denote by $f^h\eta$ its horizontal lift to X (i.e., the vector field on X orthogonal to the fibers of f and projected to η).

Consider the one-dimensional vector bundle $\det T_vX = \Lambda^kT_vX$. Since it is a subbundle of Λ^kTX, the Lie derivative $L_\zeta V$ is defined for a section V of the bundle $\det T_vX$ in the direction of an arbitrary vector field ζ on X. The Riemannian structure on X induces a Riemannian structure (scalar product in fibers) on bundles Λ^iTX and, in particular, a Riemannian structure on $\det T_vX$.

Lemma 2. *There exists a unique horizontal* 1*-form* α *on* X *such that*

$$L_{\eta'}V = (\alpha, \eta')V \tag{3.1}$$

for an arbitrary vector field η *on* Y*, its horizontal lift* $\eta' = f^h\eta$*, and a* (*local*) *section* V *of the bundle* $\det T_vX$ *such that* $\langle V, V\rangle = 1$.

We will call the form α the *fiber expansion factor under transition in horizontal direction.*

Proof. The local one-parameter group of diffeomorphisms of X generated by the vector field η' takes fibers to fibers. Therefore, for fixed sections η and V, there is a uniquely defined real-valued function $c\colon X\to\mathbb{R}$ such that $L_{\eta'}V=c(x)V$. The proportionality factor $c(x)$ is independent of V, since the field V is defined locally uniquely up to sign and, under a sign change in V, both sides of the above equality change their signs. Thus, the function $c(x)$ depends only on the vector field η, which can be regarded as a parameter: $c=c(x;\eta)$. Moreover, the value of c at a point x depends only on the 1-jet of η' at x, i.e., on the 1-jet of η at the point $f(x)$.

Let us now fix a point $y\in Y$ and restrict $c(x;\eta)$ to the fiber $Z=f^{-1}(y)\subset X$. We obtain an $\mathbb{R}$-linear map $c_y\colon T_yY\otimes_{\mathbb{R}}(O_y/\mathfrak{m}^2O_y)\to C^\infty(Z)$, which assigns to a 1-jet of η at the point y a real-valued function on Z. Here, O_y is the local ring of germs of smooth functions on Y at the point y and $\mathfrak{m}$ is the maximal ideal of the ring O_y consisting of germs of functions that vanish at y.

If $\eta(y)=0$, then η' vanishes on the whole fiber Z and the flow along η' fixes this fiber; hence, $L_{\eta'}V\big|_x=0$ and $c(x;\eta)=0$ for all $x\in Z$. Thus, the restriction of c_y to $T_yY\otimes_{\mathbb{R}}(\mathfrak{m}O_y/\mathfrak{m}^2O_y)$ vanishes. Therefore, c_y goes through $T_yY\otimes_{\mathbb{R}}(O_y/\mathfrak{m}O_y)=T_yY\otimes_{\mathbb{R}}\mathbb{R}=T_yY$, and $c_y=h(x)(\beta,\eta(y))$ for some smooth function $h\colon Z\to\mathbb{R}$ and $\beta\in T^*_yY$. This equality can also be written as $c_y=(\alpha_y(x),\eta'(x))$, where $\alpha_y=h\cdot f^*\beta$ is a uniquely defined horizontal form over Z, i.e., a section of the bundle $T^*_hX|_Z$. Since the point y was chosen arbitrarily, we conclude that there exists a unique section α of the bundle T^*_hX such that $c(x,\eta)=(\alpha(x),\eta'(x))$, which was to be proved. ☐

Theorem 2. *For a surjective submersion $f:X\to Y$, map* (0.3) *is a morphism from equation* (0.1) *if and only if the following conditions hold* (*for the vector field Ξ and the metric on Y defined as in Theorem* 1):

(a) $\langle f^*\omega,f^*\omega'\rangle=f^*\langle\omega,\omega'\rangle$ *for any* $\omega,\omega'\in T^*Y$;

(b) $L_{\eta'}V=(-1)^k(\langle\xi,\eta'\rangle-\langle\Xi,\eta\rangle)V$ *for an arbitrary vector field η on Y, its horizontal lift $\eta'=f^h\eta$, and a* (*local*) *section V of the bundle* $\det T_vX$ *such that* $\langle V,V\rangle=1$.

Proof. Recall that a Euclidean scalar product on a finite-dimensional vector space U induces a scalar product on exterior powers Λ^iU as well as on U^* and Λ^iU^*, which, in turn, defines a canonical isomorphism between Λ^iU and Λ^iU^*. We will denote this isomorphism by a tilde over a letter that denotes a polyvector or a polycovector, so that $v(u)=\langle\widetilde{u},v\rangle=\langle u,\widetilde{v}\rangle$, where $\widetilde{v}\in\Lambda^iU$ and $\widetilde{u}\in\Lambda^iU^*$ for any $u\in\Lambda^iU$ and $v\in\Lambda^iU^*$. If there is also an orientation on U, then the Hodge operator $*\colon\Lambda^iU\to\Lambda^{\dim U-i}U$ is defined, and $\langle u,v\rangle=*(u\wedge *v)$ for all $u,v\in\Lambda^iU$.

Implication 1 ⇒ 2. Let $f:X\to Y$ specify a morphism from (0.1) to (0.4). By Theorem 1, the following equality holds for any function $v\colon Y\to\mathbb{R}$ and $u=f^*v$:

$$\Delta u+L_\xi u=f^*(\Delta v+L_\Xi v). \tag{3.2}$$

(a) For $x\in X$, $y=f(x)$, and an arbitrary form $\omega\in T^*Y$, consider a function $q\colon Y\to\mathbb{R}$ such that $q|_y=0$ and $\mathrm{d}q|_y=\omega|_y$ and define $v=q^2/2$. Then

$$\Delta v+L_\Xi v|_y=\langle\omega,\omega\rangle_y,\qquad \Delta f^*v+L_\xi f^*v|_x=\langle f^*\omega,f^*\omega\rangle_x.$$

Since the point x is chosen arbitrarily, we obtain from (3.2)

$$\forall\omega\in T^*Y\quad f^*\langle\omega,\omega\rangle\equiv\langle f^*\omega,f^*\omega\rangle. \tag{3.3}$$

Passing from quadratic forms to their polarizations, we obtain condition (a) of the theorem.

Remark. In the general case, the identity $Df^*=f^*D'$ for differential operators D on X and D' on Y implies the identity $\sigma'=f_*\sigma$ for the principal symbols σ and σ' of the operators D and D', respectively (here, $f_*\sigma$ is defined by the formula $(f_*\sigma)(\omega)=\sigma(f^*\omega)$ for $\omega\in T^*Y$). In our situation, $\sigma'(\omega)=\|\omega\|^2$ and $\sigma(\omega')=\|\omega'\|^2$, and we obtain equality (3.3).

(b) Let us first consider the case when both manifolds X and Y are oriented. Then, the Laplace – Beltrami operator on functions is given by the formula $\Delta = *\mathrm{d} * \mathrm{d}$ [9].

Denote by $\rho = *1 \in \Lambda^n(T^*X)$ and $\sigma = *1 \in \Lambda^m(T^*Y)$ volume forms on X and Y, respectively. Define $\sigma' = f^*\sigma$ and $\chi = *\sigma'$. Condition (3.3) implies that $\langle\chi,\chi\rangle = \langle\sigma',\sigma'\rangle = 1$, the restriction of χ to each fiber $f^{-1}(y)$ is a volume form on the fiber, and $\sigma' \wedge \chi = \rho$.

Apply to both sides of identity (3.2) the Hodge operator and write in detail the left- and right-hand sides of the obtained equality, taking into account that the equality $*(f^*\omega) = f^*(*\omega) \wedge \chi$ holds for any differential form ω on Y:

$$*\mathrm{d}u = *(f^*\mathrm{d}v) = f^*(*\mathrm{d}v) \wedge \chi,$$

$$*(\Delta u + L_\xi u) = \mathrm{d} * \mathrm{d}u + \tilde{\xi} \wedge *\mathrm{d}u = f^*(\mathrm{d} * \mathrm{d}v) \wedge \chi + (-1)^{m-1} f^*(*\mathrm{d}v) \wedge \mathrm{d}\chi + \tilde{\xi} \wedge f^*(*\mathrm{d}v) \wedge \chi,$$

$$*f^*(\Delta v + L_\Xi v) = f^*(*\Delta v + *(L_\Xi v)) \wedge \chi = f^*\left(\mathrm{d} * \mathrm{d}v + \tilde{\Xi} \wedge *\mathrm{d}v\right) \wedge \chi.$$

Equating the last two expressions, we find that the form

$$\omega = \mathrm{d}\chi + \left(\tilde{\xi} - f^*\tilde{\Xi}\right) \wedge \chi$$

satisfies the condition

$$\forall v \in C^\infty(Y) \quad f^*(*\mathrm{d}v) \wedge \omega = 0.$$

Since the value at a fixed point $x \in X$ of an arbitrary horizontal $(m-1)$-form can be written as the value at the same point of the form $f^*(*\mathrm{d}v)$ for an appropriate function $v \colon Y \to \mathbb{R}$, the exterior product of ω and any horizontal $(m-1)$-form on X is zero. This condition can also be written in the form

$$i_{\tilde{\chi}}\omega = 0$$

for the vertical polyvector field $\tilde{\chi} \in \Lambda^k(T_v X)$ dual to χ.

Now, let η be an arbitrary vector field on Y, and let η' be its lift to a horizontal vector field on X. By (3.1), a unit volume element of the fiber $V = \tilde{\chi}$ at the point $x \in X$ expands under a shift in the direction of the field η' at the rate of (α, η'); i.e., $L_{\eta'}\tilde{\chi} = (\alpha, \eta')\tilde{\chi}$. Since $(\chi, \tilde{\chi}) = 1$, we obtain $(\alpha, \eta') = \left(\chi, L_{\eta'}\tilde{\chi}\right) = -\left(L_{\eta'}\chi, \tilde{\chi}\right) = -\left(\mathrm{d}\left(i_{\eta'}\chi\right) + i_{\eta'}\mathrm{d}\chi, \tilde{\chi}\right)$. However, η' is a horizontal vector field and χ is a vertical form; hence, $i_{\eta'}\chi = 0$. Thus,

$$(\alpha, \eta') = -i_{\tilde{\chi}} i_{\eta'} \mathrm{d}\chi = (-1)^{k+1} i_{\eta'} i_{\tilde{\chi}} \mathrm{d}\chi = (-1)^k i_{\eta'}\left(\tilde{\xi} - f^*\tilde{\Xi}\right) = (-1)^k \left(\langle\xi, \eta'\rangle - \langle\Xi, \eta\rangle\right),$$

which completes the proof of condition (b) in the oriented case.

Let us return to the general case, when X and Y are not necessary oriented. Note that both identity (3.2) and condition (b) of the theorem are local and do not depend on the local choice of orientation. Therefore, condition (b) of the theorem holds independently of the presence of orientation on X and Y.

Implication 2 ⇒ 1. Assume that conditions (a) and (b) of the theorem are satisfied. Repeating the argument of the first part of the proof in the reverse direction, we obtain (3.2). Hence, by Theorem 1, f specifies a morphism from equation (0.1) to equation (0.4). □

In the case $\xi = 0$, the above theorem takes the following form.

Theorem 3. *For a surjective submersion* $f : X \to Y$, *map* (0.3) *is a morphism from equation* (0.2) *if and only if the following conditions hold*:

(a) $\langle f^*\omega, f^*\omega'\rangle = f^*\langle \omega, \omega'\rangle$ *for any* $\omega, \omega' \in T^*Y$;

(b) $\alpha = f^*\beta$ *for some* 1*-form* β *on* Y; *i.e., the expansion factor of a fiber in the direction of the vector field* η' *is independent of the point of the fiber.*

In this case, the factor-equation has form (0.4) *and* $\tilde{\Xi} = (-1)^{k+1}\beta$.

4. LOCALLY TRIVIAL BUNDLES

In what follows, we additionally restrict the class of morphisms under consideration by the following condition:

$$\text{the projection } f\colon X \to Y \text{ is a locally trivial bundle.}$$

In this case, any piecewise smooth path $\gamma\colon [0,1] \to Y$ specifies a diffeomorphism Φ_γ of the fiber over the beginning of the path $\gamma(0)$ to the fiber over the end of the path $\gamma(1)$: the point $x \in f^{-1}(\gamma(0))$ corresponds to the end of the horizontal curve that is projected to γ and starts at x. We will call this diffeomorphism the *transition of the fiber along* γ.

Let A and B be two closed subsets of X. We will say that A and B are parallel if $d(x,B)$ is independent of the choice of the point $x \in A$ and $d(y,A)$ is independent of the choice of the point $y \in B$. Here, $d(x,B)$ is defined as the infimum of distances $d(x,y)$ from x to $y \in B$. If A and B are parallel, then the Hausdorff distance $d(A,B)$ is equal to $d(x,B)$ and $d(y,A)$ for any $x \in A$ and $y \in B$.

Theorem 4. *A locally trivial bundle* $f : X \to Y$ *specifies a morphism from equation* (0.2) *if and only if the following conditions hold*:

(a) *any two fibers of the bundle* f *are parallel*;

(b) *transitions along piecewise smooth paths on* Y *change the volume of the fiber proportionally* (*i.e., the expansion factor depends only on the path and does not depend on the choice of the point of the fiber*).

Proof. It is sufficient to prove that, for a locally trivial bundle f, conditions (a) and (b) of this theorem are equivalent to conditions (a) and (b) of Theorem 3.

Assume that f satisfies conditions (a) and (b) of Theorem 3. Fix arbitrary $y_0, y_1 \in Y$ and define $Z_i = f^{-1}(y_i)$. Let $\gamma\colon [0,1] \to Y$ be a piecewise smooth curve in Y connecting the points y_0 and y_1. It can be lifted to a piecewise smooth horizontal curve $\gamma'\colon [0,1] \to X$ with the beginning at an arbitrary point $x_0 \in Z_0$ and the end at Z_1. From condition (a) of Theorem 3, we obtain

$$d(x_0, Z_1) \le l(\gamma') = \int_0^1 |\gamma'_s|\,\mathrm{d}s = \int_0^1 |\gamma_s|\,\mathrm{d}s = l(\gamma).$$

Passing to the infimum over all such curves γ, we obtain $d(x_0, Z_1) \le d(y_0, y_1)$. On the other hand, for any point $x_1 \in Z_1$ and a piecewise smooth curve $\beta\colon [0,1] \to X$ connecting x_0 and x_1, we have

$$l(\beta) = \int_0^1 |\beta_s|\,\mathrm{d}s \ge \int_0^1 |f_*\beta_s|\,\mathrm{d}s = l(f\beta) \ge d(y_0, y_1).$$

Passing to the infimum over all such curves β, we obtain $d(x_0, x_1) \ge d(y_0, y_1)$. Thus, $d(x_0, Z_1) = d(y_0, y_1)$. Similarly, $d(x_1, Z_0) = d(y_1, y_0)$ for any point $x_1 \in Z_1$. This proves that the fibers $Z_0 = f^{-1}(y_0)$ and $Z_1 = f^{-1}(y_1)$ are parallel for any $y_0, y_1 \in Y$.

Integrating condition (b) of Theorem 3, we find that the transition $\Phi_\gamma\colon Z_0 \to Z_1$ along an arbitrary piecewise smooth curve $\gamma\colon [0,1] \to Y$, where $Z_i = f^{-1}(\gamma(i))$, changes all volumes by the same factor:

$$\operatorname{volume}(\Phi_\gamma U) = \exp\left((-1)^{k+1}\int\limits_\gamma \tilde{\Xi}\right)\operatorname{volume}(U) \tag{4.1}$$

for any measurable $U \subset Z_0$.

The converse implication (from the conditions of this theorem to conditions (a) and (b) of Theorem 3) is obvious. □

Corollary 1. *Let a local bundle $f : X \to Y$ specify a geometric morphism from equation* (0.2) *to equation* (0.4)*. The holonomy group of the bundle f* (*given by planes orthogonal to fibers for the sake of connectedness on X*) *preserves volume on a fiber if and only if the form $\tilde{\Xi}$ is exact.*

Proof. Identity (4.1) implies that the holonomy group f preserves volume on a fiber if and only if the integral of the form $\tilde{\Xi}$ along any closed curve is zero. This condition is equivalent to the exactness of $\tilde{\Xi}$. □

Corollary 2. *Let a local bundle $f : X \to Y$ specify a geometric morphism from equation* (0.2)*. If a fiber of f has finite volume* (*in particular, if a fiber is compact*)*, then the factor-equation has the form $D_t v = \varphi^{-1}\operatorname{div}(\varphi\nabla v)$ for some smooth function $\varphi\colon Y \to \mathbb{R}_+$.*

Proof. If a fiber f has finite volume, then transitions along closed paths on Y must preserve this volume. Hence, by Corollary 1, $\tilde{\Xi} = \mathrm{d}\psi$ for some smooth function $\psi\colon Y \to \mathbb{R}$. Define $\varphi = \exp(\psi)$; then, $\Delta v + L_\Xi v = \varphi^{-1}(\varphi\Delta v + (\mathrm{d}\varphi, \nabla v)) = \varphi^{-1}\operatorname{div}(\varphi\nabla v)$ and equation (0.4) takes the required form. □

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REFERENCES

1. M. F. Prokhorova, *Factorization of Partial Differential Equations: Structure of the Parabolic Equations Category*, Preprint No. 10/2005 (St. Petersb. Dep. Steklov Inst. Math., Russ. Acad. Sci., 2005).
2. M. F. Prokhorova, *Modeling of the Heat Equation and Stefan Problem* (IMM UrO RAN, Yekaterinburg, 2000), Available from VINITI, No. 347-V00 [in Russian].
3. M. F. Prokhorova, "The structure of the category of parabolic equations," http://arxiv.org/pdf/math/0512094v5.pdf. Submitted May 29, 2009.
4. P. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 1986; Mir, Leningrad, 1989).
5. A. M. Vinogradov, "Geometry of nonlinear differential equations," in *Results in Science and Technology* (VINITI, Moscow, 1980), Ser. Geometry Problems, Vol. 11, pp. 89–134 [in Russian].
6. A. M. Vinogradov, "Category of nonlinear differential equations," in *Global Analysis. Studies and Applications I*, Ed. by Yu.G. Borisovich and Yu.E. Gliklikh (Springer, Berlin, 1984), Ser. Lecture Notes in Mathematics, Vol. 1108, pp. 77–102.
7. A. V. Bocharov, A. M. Verbovetskii, A. M. Vinogradov, et al., *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics* (Faktorial, Moscow, 1997; Amer. Math. Soc., Providence, RI, 1999).
8. M. Hirsch, *Differential Topology* (Springer, New York, 1976; Mir, Moscow, 1979).
9. F. Warner, *Foundations of Differentiable Manifolds and Lie Groups* (Springer, New York, 1971; Mir, Moscow, 1987).

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