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Nikol'skii Inequality Between the Uniform Norm and L_q-Norm with Ultraspherical Weight of Algebraic polynomials on a Segment

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Dedicated to Professor E. B. Saff on the occasion of his 70th birthday

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Abstract We study the Nikol'skii inequality for algebraic polynomials on the segment $[-1,1]$ between the uniform norm and the norm of the space L_q^{ϕ}, $1 \le q < \infty$, with the ultraspherical weight $\phi(x) = \phi^{(\alpha,\alpha)}(x) = (1-x^2)^{\alpha}$, $\alpha \ge -1/2$. We prove that the polynomial with unit leading coefficient that deviates least from zero in the space L_q^{ψ} with the Jacobi weight $\psi(x) = \phi^{(\alpha+1,\alpha)}(x) = (1-x)^{\alpha+1}(1+x)^{\alpha}$ is an extremal polynomial in the Nikol'skii inequality. To prove the result, we use the generalized translation.

Keywords Algebraic polynomial · Nikol'skii inequality · Polynomials that deviate least from zero · Generalized translation

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1 Introduction. Statement of problems

1.1 Nikol'skii inequality on a segment with weight

Let a function v be summable, nonnegative, and almost everywhere nonzero on $(-1,1)$; we will call such function a weight. Denote by $L_q^v = L_q^v(-1,1)$, $1 \le q < \infty$, the space of (real-valued) functions f measurable on $(-1,1)$ and

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such that the product $|f|^q \upsilon$ is summable over $(-1,1)$. This is a Banach space with respect to the norm

$$\|f\|_{L_q^\upsilon(-1,1)} = \left(\int_{-1}^{1} |f(x)|^q \upsilon(x)\, dx \right)^{1/q}, \qquad f \in L_q^\upsilon(-1,1).$$

Along with $L_q^\upsilon(-1,1)$, consider the space $C = C[-1,1]$ of functions continuous on $[-1,1]$ with the uniform norm

$$\|f\|_{C[-1,1]} = \max\{|f(x)| \colon x \in [-1,1]\}.$$

Denote by $M(n,\upsilon) = M(n,\upsilon)_q$ the best (i.e., the least possible) constant in the inequality

$$\|p\|_{C[-1,1]} \le M(n,\upsilon)\, \|p\|_{L_q^\upsilon(-1,1)}, \qquad p \in \mathscr{P}_n, \tag{1.1}$$

on the set $\mathscr{P}_n = \mathscr{P}_n(\mathbb{R})$ of algebraic polynomials (in one variable) of degree (at most) n with real coefficients.

In the present paper, we study inequality (1.1) with the ultraspherical weight

$$\phi(x) = \phi^{(\alpha)}(x) = (1-x^2)^\alpha, \qquad \alpha > -1. \tag{1.2}$$

In what follows, the space L_q^ϕ with weight (1.2) will be denoted by L_q^α. Thus, in the present paper, we study the inequality

$$\|p\|_C \le M_n\, \|p\|_{L_q^\alpha}, \qquad p \in \mathscr{P}_n, \tag{1.3}$$

with the best constant $M_n = M(n, \phi^{(\alpha)})_q$.

Inequalities (1.1) and (1.3) are analogs of Nikol'skii inequality [29] for algebraic polynomials on a segment. Such inequalities and more general inequalities between the uniform norm and integral norms with weights (especially with Jacobi weights) of derivatives of algebraic polynomials and the polynomials themselves were studied by A.A. Markov, V.A. Markov, S.N. Bernstein, M.K. Potapov, I.K. Daugavet, S.Z. Rafal'son, V.I. Ivanov, S.V. Konyagin, B. Bojanov, P.Yu. Glazyrina, I.E. Simonov, and many others; see monographs [30,37, 28,11,31,26], papers [10,13,21,22,24,17,16,18,33–35], and references therein. In particular, [13] contains the order of behavior of the best constant $M(n,\phi^{(\alpha)})_q$ in (1.3) with respect to n as $n \to \infty$:

$$M(n,\phi^{(\alpha)})_q \asymp n^\gamma, \qquad \gamma = \frac{2(\alpha+1)}{q}.$$

At present, a great number of works are devoted to related sharp inequalities for trigonometric polynomials. Such inequalities were studied by S.N. Bernstein, M. Riesz, G. Szegö, A. Zygmund, S.B. Stechkin, A.P. Calderon, G. Klein, L.V. Taikov, E.B. Saff, T. Sheil-Small, P. Nevai, E.A. Storozhenko, V.G. Krotov, P. Osvald, V.I. Ivanov, S.V. Konyagin, A.I. Kozko, Q.I. Rahman, G. Schmeisser, N.P. Korneichuk, V.F. Babenko, A.A. Ligun, V.A. Kofanov, S.A.

Pichugov, A.G. Babenko, V.V. Arestov, P.Yu. Glazyrina, and many others; see monographs [42,28,26,11,31], papers [12,39,32,21,22,1,2,6,7,4,3], and references therein.

Consider an auxiliary inequality

$$|p(1)| \le D_n \, \|p\|_{L_q^\alpha}, \qquad p \in \mathscr{P}_n, \tag{1.4}$$

with the best constant $D_n = D(n, \phi^{(\alpha)})_q$. This inequality is also of independent interest. It is clear that $D_n \le M_n$. We show below that, in fact, $D_n = M_n$ at least for $\alpha \ge -1/2$.

The aim of this paper is to study the properties of extremal polynomials in inequalities (1.3) and (1.4), i.e., of polynomials $\rho_n \in \mathscr{P}_n,\ \rho_n \not\equiv 0$, for which these inequalities turn into equalities. In particular, we will study the uniqueness of extremal polynomials. It is clear that, if a polynomial ρ_n is extremal, then the polynomial $c\rho_n$ for any constant $c \ne 0$ is also extremal. If ρ_n is an extremal polynomial in inequality (1.4) and any other extremal polynomial has the form $c\rho_n,\ c \ne 0$, then we will say that ρ_n is the *unique* extremal polynomial in (1.4). Since weight (1.2) is even, along with the polynomial $\rho_n(x)$, the polynomial $\rho_n(-x)$ is also extremal in (1.3). Therefore, in this case, the polynomial ρ_n will be called *unique extremal* in (1.3) if any other extremal polynomial has the form $c\rho_n(\pm x),\ c \ne 0$.

1.2 Polynomials that deviate least from zero

Given the weight

$$\psi(x) = \phi(x)(1-x) = (1-x)^{\alpha+1}(1+x)^\alpha,$$

a parameter $q,\ 1 \le q < \infty$, and integer $n \ge 1$, denote by $\varrho_n = \varrho_{n,\psi,q}$ the polynomial of degree n with unit leading coefficient that deviates least from zero in the space L_q^ψ. The polynomial $\varrho_n = \varrho_{n,\psi,q}$ is a solution of the problem

$$\min\{\|p_n\|_{L_q^\psi} : p_n \in \mathscr{P}_n^1\} = \|\varrho_n\|_{L_q^\psi}, \tag{1.5}$$

where $\mathscr{P}_n^1$ is the set of polynomials $p_n(x) = x^n + \sum_{k=0}^{n-1} a_k x^k$ of degree n with leading coefficient 1.

Solution of problem (1.5) for $q = 2$ is well-known. In this case, the Jacobi polynomial $R_n^{(\alpha+1,\alpha)}$ of degree n divided by its leading coefficient solves the problem; see the next section for details. Problem (1.5) for $q = 1$ reduces to studying a system of n polynomial equations in n variables which can be solved at least for small n immediately or by finding a Gröbner basis; see, for example, [14, Subsect. 3.3].

1.3 Main result

The following statement is the main results of the present paper.

Theorem 1 *For $\alpha \geq -1/2,\ 1 \leq q < \infty$, and $n \geq 1$, the following statements are valid.*

(1) *The best constants in inequalities* (1.3) *and* (1.4) *coinside*:

$$M_n(\phi^{(\alpha)})_q = D_n(\phi^{(\alpha)})_q. \tag{1.6}$$

(2) *The polynomial ϱ_n that deviates least from zero with respect to the norm of the space L_q^ψ is the unique extremal polynomial in inequality* (1.4).

(3) *The polynomial ϱ_n is an extremal polynomial in inequality* (1.3). *This polynomial is unique extremal in the case $\alpha > -1/2$.*

Theorem 1 reduces the problem of studying inequality (1.3) to studying problem (1.5), which, to our mind, is essentially simpler.

For

$$\alpha = \frac{m-3}{2}, \qquad \text{where } m \text{ is integer}, \quad m \geq 3, \tag{1.7}$$

the statement of the theorem was proved in the author's paper [5] in parallel with studying the Nikol'skii inequality between the uniform norm and the L_q-norm of algebraic polynomials on the unit sphere of the Euclidean space $\mathbb{R}^m,\ m \geq 3$. For $q = 1$ and values (1.7) of the parameter α, the statement of the theorem (except for the uniqueness of an extremal polynomial in inequality (1.3)) was proved even earlier in [14].

The case $\alpha = -1/2$ is special in this research area. The weight $\phi(x) = (1-x^2)^{-1/2}$ is called the Chebyshev weight. Inequality (1.3) for the Chebyshev weight can be written as the classical Nikol'skii inequality on the set $\mathscr{T}_n$ of trigonometric polynomials of degree (at most) n. Denote by $\widetilde{L}_q,\ 1 \leq q < \infty$, the space of 2π-periodic real-valued measurable functions F such that the function $|F|^q$ is summable over $(-\pi, \pi)$. This is a Banach space with respect to the norm

$$\|F\|_{\widetilde{L}_q} = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} |F(\eta)|^q \, d\eta \right)^{1/q}. \tag{1.8}$$

For a function $f \in L_q^{-1/2}$, we have

$$\|f\|_{L_q^{-1/2}} = \left(\int_{-1}^{1} |f(x)|^q (1-x^2)^{-1/2} dx \right)^{1/q} = \left(\int_{0}^{\pi} |f(\cos\eta)|^q d\eta \right)^{1/q}. \tag{1.9}$$

The formula $F(\eta) = f(\cos\eta),\ \eta \in \mathbb{R}$, establishes a one-to-one correspondence between the space $L_q^{-1/2}$ and the subspace of even functions from $\widetilde{L}_q$; moreover, by (1.9), $\|f\|_{L_q^{-1/2}} = (\pi/2)^{1/q} \|F\|_{\widetilde{L}_q}$. Along with $\widetilde{L}_q$, consider the space $C_{2\pi}$ of 2π-periodic functions continuous on the whole real line with the uniform norm. Denote by $\widetilde{M}(n) = \widetilde{M}(n)_q$ the best (i.e., the least possible) constant in the inequality

$$\|F_n\|_{C_{2\pi}} \leq \widetilde{M}(n)_q \, \|F_n\|_{\widetilde{L}_q}, \qquad F_n \in \mathscr{F}_n, \tag{1.10}$$

on the set of trigonometric polynomials of given degree $n \geq 1$. Obviously, the best constants in (1.3) for the Chebyshev weight and in (1.10) are related by the inequality $M\left(n, \phi^{(-1/2)}\right)_q \leq (2/\pi)^{1/q} \widetilde{M}(n)_q$. It is not hard to show that, in fact,

$$M\left(n, \phi^{(-1/2)}\right)_q = (2/\pi)^{1/q} \; \widetilde{M}(n)_q.$$

Not many results on sharp inequalities (1.1), (1.3), and (1.10) are known. Probably, Jackson [23] was the first who studied inequality (1.10). At present, the case $q = 1$ is the most completely studied. S.B. Stechkin showed (see [38,40]) that, for the constant $\widetilde{M}(n) = \widetilde{M}(n)_1$, there exists a finite limit $c = \lim\limits_{n\to\infty} \widetilde{M}(n)/n$. Taikov [38] (see also [40]) obtained close upper and lower estimates for the quantity c. To the present, the best results about the constant $\widetilde{M}(n)$ have been obtained by Babenko, Kofanov, and Pichugov [9] and Gorbachev [20] (see also [19]). Gorbachev, in addition, established [20,19] the relation between this problem and other extremal problems of function theory.

Lupas [27], in particular, obtained sharp inequality (1.1) for the Jacobi weight $\upsilon(x) = (1-x)^\alpha (1+x)^\beta$ with $\alpha, \beta \geq -1/2$ for $q = 2$ (i.e., he found the best constant and an extremal polynomial).

Glazyrina and Simonov [35] constructed a polynomial extremal in inequality (1.3) with the Chebyshev weight for $q = 1$ (in the form of a linear combination of Chebyshev polynomials of the first kind), proved its uniqueness, and showed that its uniform norm is attained at an end-point of the segment $[-1, 1]$.

Note that our method of studying does not allow us to prove the uniqueness of the extremal polynomial ϱ_n in inequality (1.3) for $\alpha = -1/2$, i.e., for the Chebyshev weight.

2 Pointwise Nikol'skii inequality for algebraic polynomials on a segment

The aim of this section is to study an analog of inequality (1.4) for arbitrary weight:

$$|p_n(1)| \leq D(n, \upsilon)_q \, \|p_n\|_{L_q^\upsilon}, \qquad p_n \in \mathscr{P}_n. \tag{2.1}$$

2.1 Pointwise Nikol'skii inequality on a segment with arbitrary weight

Along with inequality (2.1), we are interested in the more general pointwise inequality

$$|p_n(z)| \leq D(n, \upsilon, z)_q \, \|p_n\|_{L_q^\upsilon}, \qquad p_n \in \mathscr{P}_n, \tag{2.2}$$

for $z \in \mathbb{C}$. To the present time, a large number of studies have been devoted to inequalities (2.2) and (1.1), see [26, Sect. 6.1], [28, Ch. 4], [37, Sect. 7.71].

Inequalities (2.2) and (1.1) are studied most completely for $q = 2$; see [30, Part VI, Sect. 12], [37, Sect. 7.71]. Let $\{p_n^{\upsilon}\}_{n=0}^{\infty}$ be a system of polynomials orthonormal on $(-1, 1)$ with weight υ. Then, for $z \in [-1, 1]$ and $q = 2$, the square of the best constant in (2.2) satisfies the formula (see, for example, [37, Sect. 3.1, Thm. 3.1.3])

$$(D(n, \upsilon, z)_2)^2 = \sum_{k=0}^{n} (p_k^{\upsilon}(z))^2, \tag{2.3}$$

and the polynomial

$$\rho_n(x) = \sum_{k=0}^{n} p_k^{\upsilon}(z) p_k^{\upsilon}(x) \tag{2.4}$$

is extremal. As a consequence of (2.3), the following formula is valid:

$$(D(n, \upsilon)_2)^2 = \max\left\{\sum_{k=0}^{n} (p_k^{\upsilon}(z))^2 : z \in [-1, 1]\right\}. \tag{2.5}$$

The Jacobi polynomials $\{p_n^{(\alpha,\beta)}\}$, which are orthonormal on $(-1, 1)$ with Jacobi weight

$$\phi^{(\alpha,\beta)}(x) = (1 - x)^{\alpha}(1 + x)^{\beta}, \tag{2.6}$$

have the following property for $\alpha \geq \beta \geq -1/2$ (see, for example, [37, Sect. 7.32, Thm. 7.32.1; Sect. 4.1, formula (4.1.1)])

$$\max\{|p_k^{(\alpha,\beta)}(x)| : \ x \in [-1, 1]\} = |p_k^{(\alpha,\beta)}(1)|, \qquad k \geq 0.$$

Hence, by (2.5) and (2.3), for Jacobi weight (2.6) with $\alpha \geq \beta \geq -1/2$ and $q = 2$, the constants in inequalities (1.1) and (2.1) coincide:

$$M\left(n, \phi^{(\alpha,\beta)}\right)_2 = D\left(n, \phi^{(\alpha,\beta)}\right)_2$$

and the square of their common value is (see, for example, [30, Part VI, Sect. 12, Thm. 105]

$$\frac{1}{2^{\alpha+\beta+1}} \times \frac{\Gamma(n + \alpha + 2)\Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\alpha + 2)\Gamma(n + 1)\Gamma(n + \beta + 1)}.$$

The polynomial

$$\sum_{k=0}^{n} p_k^{(\alpha,\beta)}(1) p_k^{(\alpha,\beta)}(x) \tag{2.7}$$

is extremal in inequalities (2.1) and (1.1) for $q = 2$ and Jacobi weight (2.6) with values of the parameters $\alpha \geq \beta \geq -1/2$ (see (2.4)). These results for Jacobi weight (2.6) are contained in [27]; we give them here to illustrate general fact (2.5); they also will be useful in the further discussion.

Polynomial (2.7), up to a constant factor, coincides [37, Sect. 4.5, formulas (4.5.2), (4.5.3)] with the Jacobi polynomial $p_n^{(\alpha+1,\beta)}$, which corresponds to the weight $\phi^{(\alpha+1,\beta)}(x) = (1-x)\,\phi^{(\alpha,\beta)}(x) = (1-x)^{\alpha+1}(1+x)^{\beta}$. It is well-known that the polynomial $p_n^{(\alpha+1,\beta)}$ divided by its leading coefficient deviates least from zero in the space L_2^{υ} with the weight $\upsilon = \phi^{(\alpha+1,\beta)}$. This fact and the results of [27] contain, in particular, the statements of Theorem 1 for $q = 2$. To prove Theorem 1 for $q \neq 2$, we will use other arguments.

2.2 Characterization of a polynomial extremal in inequality (2.1)

Given a weight υ, we define on the interval $(-1, 1)$ another weight

$$w(x) = (1-x)\upsilon(x). \tag{2.8}$$

Denote by $\varrho_n = \varrho_{n,w,q}$ the polynomial of degree $n \geq 1$ with unit leading coefficient that deviates least from zero in the space L_q^w. This polynomial is the solution of the problem

$$\min\{\|p_n\|_{L_q^w} : p_n \in \mathscr{P}_n^1\} = \|\varrho_n\|_{L_q^w}. \tag{2.9}$$

The polynomial ϱ_n is characterized by the property of orthogonality of the function $|\varrho_n|^{q-1}\operatorname{sgn}\varrho_n$ to the space $\mathscr{P}_{n-1}$ (see, for example, [25, Ch. 3, Sect. 3.3, Thms. 3.3.1, 3.3.2]):

$$\int_{-1}^{1} w(x)\,p_{n-1}(x)|\varrho_n(x)|^{q-1}\operatorname{sgn}\varrho_n(x)\,dx = 0, \qquad p_{n-1} \in \mathscr{P}_{n-1}. \tag{2.10}$$

From this, in particular, it follows easily that all n zeros of the polynomial ϱ_n are simple and lie in the interval $(-1, 1)$.

According to the following theorem, problems (2.9) and (2.1) have identical solution. This statement for $q = 1$ has been proved in [14]. For arbitrary $q,\ 1 \leq q < \infty$, it has been proved in [5]. To make the presentation complete, we give this theorem here with proof.

Theorem 2 *For $1 \leq q < \infty$ and $n \geq 1$, the polynomial ϱ_n of degree n with unit leading coefficient that deviates least from zero in the space L_q^w with weight* (2.8) *is the unique extremal polynomial in inequality* (2.1)*.*

Proof Let ϱ_n be the polynomial of degree n that deviates least from zero in the space L_q^w. For an arbitrary polynomial $p_n \in \mathscr{P}_n$, we have

$$\int_{-1}^{1} p_n(x)\,\upsilon(x)|\varrho_n(x)|^{q-1}\operatorname{sgn}\varrho_n(x)\,dx$$

$$= \int_{-1}^{1} r_{n-1}(x)(1-x)\upsilon(x)|\varrho_n(x)|^{q-1}\operatorname{sgn}\varrho_n(x)dx$$

$$+ p_n(1) \int_{-1}^{1} \upsilon(x)|\varrho_n(x)|^{q-1} \operatorname{sgn} \varrho_n(x) dx,$$

where

$$r_{n-1}(x) = \frac{p_n(x) - p_n(1)}{1 - x}$$

is a polynomial of degree $n-1$. This, by (2.10), implies the equality

$$\begin{aligned} & \int_{-1}^{1} p_n(x)\upsilon(x)|\varrho_n(x)|^{q-1} \operatorname{sgn} \varrho_n(x) dx \\ & = p_n(1) \int_{-1}^{1} \upsilon(x)|\varrho_n(x)|^{q-1} \operatorname{sgn} \varrho_n(x) dx. \end{aligned} \tag{2.11}$$

Let us determine the sign of the integral

$$I(n,q) = \int_{-1}^{1} \upsilon(x)|\varrho_n(x)|^{q-1} \operatorname{sgn} \varrho_n(x) dx.$$

Substituting the polynomial $p_n = \varrho_n$ into (2.11), we obtain the equality

$$\int_{-1}^{1} \upsilon(x)|\varrho_n(x)|^{q} dx = \varrho_n(1) \int_{-1}^{1} \upsilon(x)|\varrho_n(x)|^{q-1} \operatorname{sgn} \varrho_n(x) dx. \tag{2.12}$$

All zeros of the polynomial ϱ_n lie in the interval $(-1, 1)$, and its leading coefficient is positive. Hence, $\varrho_n(1) > 0$. By (2.12), we have the property

$$I(n,q) = \int_{-1}^{1} \upsilon(x)|\varrho_n(x)|^{q-1} \operatorname{sgn} \varrho_n(x)\, dx > 0.$$

Now, relation (2.11) can be written in the form

$$p_n(1) = \frac{1}{I(n,q)} \int_{-1}^{1} \upsilon(x) p_n(x)|\varrho_n(x)|^{q-1} \operatorname{sgn} \varrho_n(x) dx, \qquad p_n \in \mathscr{P}_n. \tag{2.13}$$

Using Hölder's inequality, from (2.13), we obtain the following estimate for an arbitrary polynomial $p_n \in \mathscr{P}_n$:

$$|p_n(1)| \le \frac{1}{I(n,q)} \left(\int_{-1}^{1} \upsilon(x)|p_n(x)|^q dx \right)^{\frac{1}{q}} \left(\int_{-1}^{1} \upsilon(x)|\varrho_n(x)|^q dx \right)^{\frac{q-1}{q}}. \tag{2.14}$$

For the polynomial ϱ_n, inequality (2.14) turns into an equality; this can be easily verified, for example, with the use of identity (2.13). Consequently, inequality (2.14) is inequality (2.1); moreover,

$$D(n,\upsilon)_q = \frac{\left(\|\varrho_n\|_{L_q^\upsilon}\right)^{q-1}}{I(n,q)}.$$

Based on the conditions under which Hölder's inequality turns into an equality, it is easy to conclude that, for all q, $1 \le q < \infty$, inequality (2.14) turns into an equality only for polynomials $c\varrho_n$, where $c \in \mathbb{R}$. Thus, the polynomial ϱ_n is unique extremal in inequality (2.1). Theorem 2 is proved. □

3 Generalized translation operator

The generalized translation operator plays an important role in proving Theorem 1 below. The most part of required information about its genesis, properties, and application in function theory can be found in [8].

3.1 Basic definitions and simple facts

For $1 \le q < \infty$ and $\alpha > -1$, denote by $\mathcal{L}_q^\alpha = \mathcal{L}_q^\alpha(-1,1)$ the space of complex-valued functions f measurable on $(-1,1)$ and such that the function $|f|^q$ is summable over $(-1,1)$ with ultraspherical weight (1.2). This is a Banach space with respect to the norm

$$\|f\|_{\mathcal{L}_q^\alpha} = \left(\int_{-1}^{1} |f(x)|^q (1-x^2)^\alpha \, dx.\right)^{1/q}.$$

In the limiting case $q = \infty$, we assume that $\mathcal{L}_q^\alpha$ is the classical space $\mathcal{L}_\infty = \mathcal{L}_\infty(-1,1)$ of (complex-valued) functions measurable and essentially bounded on $(-1,1)$ with the norm

$$\|f\|_{\mathcal{L}_\infty} = \operatorname{ess\,sup}\{|f(x)| : \ x \in (-1,1)\}.$$

The space $\mathcal{L}_2^\alpha$ is a Hilbert space with the inner product

$$(f,g) = (f,g)_{\mathcal{L}_2^\alpha} = \int_{-1}^{1} f(x)\overline{g(x)}(1-x^2)^\alpha dx, \qquad f,g \in \mathcal{L}_2^\alpha. \tag{3.1}$$

Let $R_\nu = R_\nu^\alpha$, $\nu \ge 0$, be a system of algebraic ultraspherical polynomials of degree ν orthogonal on the segment $[-1,1]$ with weight (1.2); more exactly, orthogonal with respect to inner product (3.1) and normalized by the condition $R_\nu(1) = 1$, $\nu \ge 0$ (see, for example, [37, Ch. IV]). These polynomials satisfy the recurrent relation (see, for example, [37, Ch. IV, Sects. 4.1, 4.5], [36, Ch. VII, Sect. 1])

$$R_{\nu+1}(z) = \frac{(2\nu+2\alpha+1)}{(\nu+2\alpha+1)} z R_\nu(z) - \frac{\nu}{(\nu+2\alpha+1)} R_{\nu-1}(z), \qquad \nu \geq 1,$$

$$R_0(z) = 1, \qquad R_1(z) = z.$$

Note for the future that, in the case $\alpha \geq -1/2$, ultraspherical polynomials $R_\nu = R_\nu^\alpha$ satisfy the relation (see, for example, [37, Ch. VII, Sect. 7.32, Thm. 7.32.1], [36, Ch. VII, Sect. 2, Thm. 7.1])

$$\max\{|R_\nu(x)| \colon x \in [-1,1]\} = R_\nu(1) = 1, \qquad \nu \geq 0. \tag{3.2}$$

In the case $\alpha > -1/2$ and $\nu \geq 1$, in addition to (3.2), we can assert that (see, for example, [37, Sects. 4.2, 7.31, 7.32])

$$|R_\nu(x)| < R_\nu(1) = 1, \qquad x \in (-1,1). \tag{3.3}$$

The system of ultraspherical polynomials $\{R_\nu\}_{\nu\geq 0}$ forms an orthogonal basis in the space $\mathcal{L}_2^\alpha$. Thus, an arbitrary function $f \in \mathcal{L}_2^\alpha$ is expanded into the Fourier series

$$f(x) = \sum_{\nu=0}^{\infty} f_\nu R_\nu(x), \qquad f_\nu = \frac{(f, R_\nu)}{(R_\nu, R_\nu)}. \tag{3.4}$$

For a pair of functions $f, g \in \mathcal{L}_2^\alpha$, the generalized version of Parseval's identity holds:

$$(f,g) = \sum_{\nu=0}^{\infty} \delta_\nu f_\nu \overline{g}_\nu, \qquad \delta_\nu = (R_\nu, R_\nu) = \|R_\nu\|^2_{\mathcal{L}_2^\alpha}. \tag{3.5}$$

In particular, the norm of a function $f \in \mathcal{L}_2^\alpha$ can be expressed in terms of its Fourier coefficients $\{f_\nu\}$ by Parseval's identity:

$$\|f\|^2_{\mathcal{L}_2^\alpha} = \sum_{\nu=0}^{\infty} \delta_\nu \, |f_\nu|^2. \tag{3.6}$$

The generalized translation operator with step $t \in [-1,1]$ is a linear operator Θ_t defined on functions $f \in \mathcal{L}_2^\alpha$ with Fourier series (3.4) by the relation (see, for example, [8, formula (0.5)] and references in [8])

$$\Theta_t f(x) = \sum_{\nu=0}^{\infty} f_\nu R_\nu(t) R_\nu(x). \tag{3.7}$$

Lemma 1 *The following two statements are valid.*

(1) *For $\alpha \geq -1/2$ and any $t \in [-1,1]$, the generalized translation operator Θ_t is a linear bounded operator in the space $\mathcal{L}_2^\alpha$ whose norm is* 1:

$$\|\Theta_t\|_{\mathcal{L}_2^\alpha \to \mathcal{L}_2^\alpha} = 1. \tag{3.8}$$

(2) *For $\alpha > -1/2$ and $t \in (-1,1)$, the norm of the operator Θ_t in the space $\mathcal{L}_2^\alpha$ is attained only at functions that coincide with some constant almost everywhere on* $(-1,1)$.

Proof Let $f \in \mathcal{L}_2^\alpha$. By property (3.2), the right-hand side of (3.7) is a function from $\mathcal{L}_2^\alpha$. Moreover, using Parseval's identity (3.6), we conclude that

$$\|\Theta_t f\|^2_{\mathcal{L}_2^\alpha} = \sum_{\nu=0}^{\infty} \delta_\nu \, |R_\nu(t)|^2 |f_\nu|^2 \le \sum_{\nu=0}^{\infty} \delta_\nu \, |f_\nu|^2 = \|f\|^2_{\mathcal{L}_2^\alpha}. \tag{3.9}$$

Thus, in the case $\alpha \ge -1/2$, the norm of the generalized translation operator Θ_t in the space $\mathcal{L}_2^\alpha$ for any $t \in [-1,1]$ is at most 1. Since $R_0(t) \equiv 1$, the translation operator (with any fixed step $t \in \mathbb{R}$) takes constant functions to themselves. This implies (3.8).

For $\alpha > -1/2$ and $t \in (-1,1)$, by (3.3), we have an equality in (3.9) if and only if all Fourier coefficients f_ν of the function f vanish for $\nu \ge 1$. But this means that f is constant. The proof of the lemma is complete. □

Note also that, by (3.5) and (3.7), we have

$$(\Theta_t f, g) = \sum_{\nu=0}^{\infty} \delta_\nu f_\nu \bar{g}_\nu R_\nu(t) = (f, \Theta_t g)$$

for any pair of functions $f, g \in \mathcal{L}_2^\alpha$ and any $t \in [-1,1]$. Thus, the equality

$$(\Theta_t f, g) = (f, \Theta_t g), \qquad f, g \in \mathcal{L}_2^\alpha, \tag{3.10}$$

holds, which means that the operator Θ_t is self-adjoint.

3.2 Integral representation of the generalized translation

In further arguments, we will use an integral representation of the generalized translation. This representation is based on the so-called product formula for ultraspherical polynomials. For $\alpha > -1/2$, integer $\nu \ge 0$, and any real θ and η, the product formula in the classical trigonometric form is (see, for example, [8, formula (5.1)], [41, Ch. 11, Sect. 11.5])

$$R_\nu(\cos\theta) R_\nu(\cos\eta) = \frac{1}{\kappa(\alpha)} \int_0^\pi R_\nu(\cos\theta \, \cos\eta + \sin\theta \, \sin\eta \, \cos\zeta)(\sin\zeta)^{2\alpha} d\zeta, \tag{3.11}$$

where

$$\kappa(\alpha) = \int_0^\pi (\sin\zeta)^{2\alpha} d\zeta = \int_{-1}^1 \left(1 - \xi^2\right)^{\alpha - 1/2} d\xi = \frac{\Gamma(\alpha + 1/2)\sqrt{\pi}}{\Gamma(\alpha + 1)}.$$

Introducing the notation $t = \cos\theta$ and $x = \cos\zeta$, we pass to the new variable of integration $\xi = \cos\zeta$ in (3.11). This results in the following formula, which is equivalent to (3.11),

$$R_\nu(t) R_\nu(x) = \frac{1}{\kappa(\alpha)} \int_{-1}^1 R_\nu\left(tx + \xi\sqrt{1-t^2}\sqrt{1-x^2}\right)\left(1 - \xi^2\right)^{\alpha - 1/2} d\xi \tag{3.12}$$

with $\alpha > -1/2$ and $-1 \le t, x \le 1$.

Substituting (3.12) into definition (3.7) of the generalized translation operator Θ_t, we obtain the following integral representation for this operator for $\alpha > -1/2$:

$$\Theta_t f(x) = \frac{1}{\kappa(\alpha)} \int_{-1}^{1} f\left(tx + \xi\sqrt{1-t^2}\sqrt{1-x^2}\right)\left(1-\xi^2\right)^{\alpha-1/2} d\xi \tag{3.13}$$

at least on the set $\mathscr{P} = \mathscr{P}(\mathbb{C})$ of all algebraic polynomials with complex coefficients.

3.3 The norm of the generalized translation operator: the case $\alpha > -1/2$

In this subsection, we will find the norm of the generalized translation operator in the spaces $\mathcal{L}_q^\alpha$, $1 \le q < \infty$, $\alpha > -1/2$. Certainly, it is of interest to describe sets of functions $f \in \mathcal{L}_q^\alpha$ at which the norm of the operator Θ_t is attained; these functions are called extremal. The values $t = \pm 1$ are of no interest because the operator Θ_1 is the unit operator, and the operator Θ_{-1} satisfies the formula $(\Theta_{-1}f)(x) = f(-x)$. For these two values of the parameter t, every function from $\mathcal{L}_q^\alpha$ is extremal. In the case $|t| < 1$, we restrict ourselves to studying extremal functions that are polynomials. This will reduce and simplify considerably the arguments, however, will be sufficient for studying the main problem of this paper.

For a (complex-valued) function f defined and measurable on a measurable set $G \subset \mathbb{R}$, we will say that f maintain sign on G if there exists a number $\zeta \in \mathbb{C}$, $|\zeta| = 1$, such that $\zeta f \ge 0$ almost everywhere on G; in this case, the number $\overline{\zeta}$ will be called the sign of the function f on G.

The following theorem is the main statement in this section.

Theorem 3 *For $\alpha > -1/2$, $1 \le q < \infty$, and any $t \in [-1, 1]$, the following statements are valid.*

(1) *The generalized translation operator Θ_t is a linear bounded operator in the space $\mathcal{L}_q^\alpha$ whose norm is* 1:

$$\|\Theta_t\|_{\mathcal{L}_q^\alpha \to \mathcal{L}_q^\alpha} = 1. \tag{3.14}$$

(2) *For $\alpha > -1/2$, $1 < q < \infty$, and $|t| < 1$, the norm of the operator Θ_t is attained at a polynomial f if and only if f is a constant.*

(3) *For $t \in (-1, 1)$, the norm of Θ_t in the space $\mathcal{L}_1^\alpha$ is attained at a polynomial f if and only if f maintains sign on* $[-1, 1]$.

Since operator (3.13) takes constant functions into themselves, to prove equality (3.14), it is sufficient to prove the inequality

$$\|\Theta_t\|_{\mathcal{L}_q^\alpha \to \mathcal{L}_q^\alpha} \le 1. \tag{3.15}$$

Further arguments will be divided into several steps. Let $\mathcal{C} = \mathcal{C}[-1, 1]$ be the space of (complex-valued) functions continuous on the segment $[-1, 1]$ with the uniform norm.

Lemma 2 *For $\alpha > -1/2$ and any $t \in [-1, 1]$, the generalized translation operator Θ_t is a linear bounded operator in the space $\mathcal{C}$ whose norm is* 1*.*

Proof Evidently, the right-hand side of (3.13), is a linear bounded operator in the space $\mathcal{C}$ and its norm is 1. Formula (3.13) holds on the set $\mathscr{P}$ of algebraic polynomials. The set $\mathscr{P}$ is dense in the space $\mathcal{C}$. Hence, we can easily conclude that the operator Θ_t can be extended by continuity to a bounded operator from the set $\mathscr{P}$ to the whole space $\mathcal{C}$; exactly formula (3.13) implements this extension. The lemma is proved. □

By (3.13), for $\alpha > -1/2$, the pointwise inequality

$$|(\Theta_t f)(x)| \le (\Theta_t |f|)(x), \qquad x \in [-1, 1], \tag{3.16}$$

is valid for $f \in \mathcal{C}$. It is clear that, if, in addition, the function f maintains sign on $[-1, 1]$, then (3.16) turns into an equality.

Lemma 3 *Let $\alpha > -1/2$ and $|t| < 1$. Then, inequality* (3.16) *turns into an equality for all $x \in (-1, 1)$ for a polynomial $f \not\equiv 0$ if and only if f maintains sign on $(-1, 1)$.*

Proof For a fixed $x \in (-1, 1)$, inequality (3.16) turns into an equality if and only if the function $f(tx + \xi\sqrt{1-t^2}\sqrt{1-x^2})$ of variable $\xi \in (-1, 1)$ has constant sign. This means that the polynomial f maintains sign on the interval

$$I(x, t) = \left(tx - \sqrt{1-t^2}\sqrt{1-x^2},\ tx + \sqrt{1-t^2}\sqrt{1-x^2}\right).$$

Let us ensure that the intervals $I(x, t)$ for $x \in (-1, 1)$ cover the interval $(-1, 1)$. For $x \in (-1, 1)$, the centers $c(x, t) = tx$ of the intervals $I(x, t)$ fill the interval $(-|t|, |t|)$. We have

$$I(t, t) = \left(2t^2 - 1,\ 1\right), \qquad c(t, t) = t^2;$$

$$I(-t, t) = \left(-1,\ 1 - 2t^2\right), \qquad c(-t, t) = -t^2.$$

Since $t^2 < |t|$, this implies that $(-1, 1) \subset \cup\{I(x, t) \colon x \in (-1, 1)\}$.

Let ρ be an arbitrary positive number satisfying the condition $|t| < \rho < 1$. We have

$$[-\rho, \rho] \subset \cup\{I(x, t) \colon x \in (-1, 1)\}.$$

The segment $[-\rho, \rho]$ is a compact set; therefore, it can be covered by a finite family of intervals $I(x, t)$. A polynomial f cannot vanish on any interval; therefore, if two intervals $I(x', t)$ and $I(x'', t)$ intersect, then the polynomial f is also of constant sign on the union of these intervals. Hence, we can easily conclude that the polynomial f maintains sign on the segment $[-\rho, \rho]$. This implies that f maintains sign on $(-1, 1)$. The lemma is proved. □

Lemma 4 *For $\alpha > -1/2$ and $q = 1$, the following three statements are valid.*

(1) *For any $t \in [-1, 1]$, the generalized translation operator Θ_t is a linear bounded operator in the space $\mathcal{L}_1^\alpha$ whose norm is* 1:

$$\|\Theta_t\|_{\mathcal{L}_1^\alpha \to \mathcal{L}_1^\alpha} = 1; \tag{3.17}$$

the norm of the operator is attained at functions $\mathcal{L}_1^\alpha$ that maintain sign on $(-1, 1)$.

(2) *For any $t \in [-1, 1]$, the generalized translation operator Θ_t in the space $\mathcal{L}_1^\alpha$ satisfies the formula*

$$\int_{-1}^{1} (\Theta_t f)(x) \left(1 - x^2\right)^\alpha dx = \int_{-1}^{1} f(x) \left(1 - x^2\right)^\alpha dx, \qquad f \in \mathcal{L}_1^\alpha. \tag{3.18}$$

(3) *For $t \in (-1, 1)$, the norm of the operator in the space $\mathcal{L}_1^\alpha$ is attained at a polynomial f if and only if f maintains sign on* $[-1, 1]$.

Proof Equality (3.10) is valid for a pair of functions $g \in \mathcal{C}$ and $f \in \mathscr{P}$. Using this equality and Lemma 2, we obtain

$$|(\Theta_t f, g)| = |(f, \Theta_t g)| \le \|f\|_{\mathcal{L}_1^\alpha} \|\Theta_t g\|_{\mathcal{C}} \le \|f\|_{\mathcal{L}_1^\alpha} \|g\|_{\mathcal{C}}.$$

Consequently, the inequality $\|\Theta_t f\|_{\mathcal{L}_1^\alpha} \le \|f\|_{\mathcal{L}_1^\alpha}$ is valid for any polynomial f. The set $\mathscr{P}$ is dense in the space $\mathcal{L}_1^\alpha$. Therefore, the operator Θ_t can be extended by continuity to a linear bounded operator in the space $\mathcal{L}_1^\alpha$; and the norm of the extended operator is at most 1. Thus, inequality (3.15) holds for $q = 1$; hence, (3.17) holds.

These arguments also allow us to assert that formula (3.10) is valid for an arbitrary pair of functions $g \in \mathcal{C}$ and $f \in \mathcal{L}_1^\alpha$. Assume that $f \in \mathcal{L}_1^\alpha$ and g is the specific function identically equal to 1. According to formulas (3.7) and (3.4), we have $\Theta_t g = g_0 = 1$. Therefore, in this case, equality (3.10) coincides with (3.18).

Formula (3.18), in particular, shows that the norm of the operator Θ_t in $\mathcal{L}_1^\alpha$ is attained at functions from $\mathcal{L}_1^\alpha$ that maintain sign on $(-1, 1)$. Let us show that any extremal polynomial has this property.

Let us verify that the inequality

$$\int_{-1}^{1} |(\Theta_t f)(x)| \left(1 - x^2\right)^\alpha dx \le \int_{-1}^{1} (\Theta_t |f|)(x) \left(1 - x^2\right)^\alpha dx \tag{3.19}$$

holds for any function $f \in \mathcal{L}_1^\alpha$. Indeed, if $f \in \mathcal{C}$, then pointwise inequality (3.16) and, hence, inequality (3.19) holds. Hence, in view of the fact that $\mathcal{C}$ is dense in $\mathcal{L}_1^\alpha$, we conclude that inequality (3.19) also holds in the space $\mathcal{L}_1^\alpha$.

Combining inequality (3.19) for a function $f \in \mathcal{L}_1^\alpha$ and equality (3.18) for the function $|f|$, we obtain

$$\begin{aligned} \|\Theta_t f\|_{\mathcal{L}_1^\alpha} = \int_{-1}^{1} |(\Theta_t f)(x)| \left(1-x^2\right)^\alpha dx \le \int_{-1}^{1} (\Theta_t |f|)(x) \left(1-x^2\right)^\alpha dx \\ = \int_{-1}^{1} |f(x)| \left(1-x^2\right)^\alpha dx = \|f\|_{\mathcal{L}_1^\alpha}. \end{aligned} \tag{3.20}$$

We arrive again at the inequality

$$\|\Theta_t f\|_{\mathcal{L}_1^\alpha} \le \|f\|_{\mathcal{L}_1^\alpha}$$

in the space $\mathcal{L}_1^\alpha$. This inequality turns into an equality for a function $f \in \mathcal{L}_1^\alpha$ if and only if inequality (3.19) turns into an equality for the function f; this is equivalent to that inequality (3.20) turns into an equality for almost all $x \in (-1, 1)$.

We see from these arguments that a polynomial f is extremal if and only if inequality (3.16) turns into an equality for this polynomial for all $x \in [-1, 1]$. Applying Lemma 3, we obtain the third statement of the lemma. □

Lemma 5 *For $\alpha > -1/2$, $1 < q < \infty$, and $t \in (-1, 1)$, the generalized translation operator Θ_t is a linear bounded operator in the space $\mathcal{L}_q^\alpha$ whose norm is* 1: $\|\Theta_t\|_{\mathcal{L}_q^\alpha \to \mathcal{L}_q^\alpha} = 1$. *The norm of the operator Θ_t is attained at a polynomial f if and only if f is a constant.*

Proof Let $f \in \mathcal{C}$. For any $x \in (-1, 1)$, using Hölder's inequality, we obtain

$$\begin{aligned} |(\Theta_t f)(x)| = \left| \frac{1}{\kappa(\alpha)} \int_{-1}^{1} f\left(tx + \xi\sqrt{1-t^2}\sqrt{1-x^2}\right) \left(1-\xi^2\right)^{\alpha-1/2} d\xi \right| \\ \le \left(\frac{1}{\kappa(\alpha)} \int_{-1}^{1} \left| f\left(tx + \xi\sqrt{1-t^2}\sqrt{1-x^2}\right)\right|^q \left(1-\xi^2\right)^{\alpha-1/2} d\xi \right)^{1/q}. \end{aligned} \tag{3.21}$$

Applying this estimate and equality (3.18) to the function $|f|^q$, we obtain

$$\|\Theta_t f\|_{\mathcal{L}_q^\alpha}^q \le \|\Theta_t(|f|^q)\|_{\mathcal{L}_1^\alpha} = \||f|^q\|_{\mathcal{L}_1^\alpha} = \|f\|_{\mathcal{L}_q^\alpha}^q.$$

Thus, functions $f \in \mathcal{C}$ satisfy the inequality $\|\Theta_t f\|_{\mathcal{L}_q^\alpha} \le \|f\|_{\mathcal{L}_q^\alpha}$. The set $\mathcal{C}$ is dense in the space L_q^α; therefore, Θ_t can be extended by continuity to a linear bounded operator in the space $\mathcal{L}_q^\alpha$ whose norm is at most 1. Thus, inequality (3.15) and, hence, equality (3.17) also holds for $1 < q < \infty$.

We have to describe polynomials at which the norm of the operator Θ_t, where $|t| < 1$, in the space L_q^α for $1 < q < \infty$ is attained. For an extremal polynomial f, Hölder's inequality in (3.21) must turn into an equality for any $x \in [-1, 1]$. For $x \in (-1, 1)$, the equality in (3.21) is attained only if the polynomial f is a constant on the segment $\left[tx - \sqrt{1-t^2}\sqrt{1-x^2},\ tx + \sqrt{1-t^2}\sqrt{1-x^2}\right]$ and, hence, on the whole segment $[-1, 1]$. Lemma 5 is proved. □

The statement of Theorem 3 is contained in Lemmas 4 and 5.

3.4 The case $\alpha = -1/2$

Let us discuss the properties of operator (3.7) for $\alpha = -1/2$. In this case, ultraspherical polynomials are the Chebyshev polynomials of the first kind:

$$R_\nu(x) = \cos(\nu \arccos x), \qquad \nu \geq 0, \qquad x \in [-1,1].$$

Setting $x = \cos\zeta$ for $\zeta \in [0,\pi]$ and $t = \cos h$ for $h \in [0,\pi]$ in (3.7), we obtain

$$T_{\cos h} f(\cos\zeta) = \sum_{\nu=0}^{\infty} f_\nu \cos(\nu h)\cos(\nu\zeta) = \frac{1}{2}(F(\zeta+h) + F(\zeta-h)), \tag{3.22}$$

where

$$F(\eta) = f(\cos\eta), \qquad \eta \in \mathbb{R}. \tag{3.23}$$

Denote by $\widetilde{\mathcal{L}}_q,\ 1 \leq q < \infty$, the space of 2π-periodic measurable complex-valued functions F such that the function $|F|^q$ is summable over $(-\pi,\pi)$. This is a Banach space with respect to norm (1.8). Formula (3.23) establishes a one-to-one correspondence between the space $\mathcal{L}_q^{-1/2}$ and the subspace of even functions from $\widetilde{\mathcal{L}}_q$; moreover, the inequality $\|f\|_{\mathcal{L}_q^{-1/2}} = (\pi/2)^{1/q}\,\|F\|_{\widetilde{\mathcal{L}}_q}$ holds, which was discussed above.

Based on formula (3.22), we define an operator $\widetilde{\Theta}_h$ for $0 \leq h \leq \pi$ in the space $\widetilde{\mathcal{L}}_q,\ 1 \leq q < \infty$, by the formula

$$(\widetilde{\Theta}_h F)(\eta) = \frac{1}{2}(F(\zeta+h) + F(\zeta-h)).$$

Evidently, the operator $\widetilde{\Theta}_h$ for any $h \in [0,\pi]$ is a linear bounded operator in the space $\widetilde{\mathcal{L}}_q$ for $1 \leq q < \infty$ and its norm is 1. As a consequence, we obtain the following statement.

Lemma 6 *For $\alpha = -1/2,\ 1 \leq q < \infty$, and any $t \in [-1,1]$, the generalized translation operator Θ_t is a linear bounded operator in the space $\mathcal{L}_q^{-1/2}$ whose norm is* 1.

In the case $\alpha = -1/2$, the situation with polynomials at which the norm of the operator Θ_t for $|t| < 1$ is attained is more complicated in comparison with the case $\alpha > -1/2$. We don't discuss this question here, because we were unable to describe polynomials extremal in inequality (1.1) for $\alpha = -1/2$ in this way.

4 Proof of Theorem 1

The best constants in inequalities (1.1) and (1.4) satisfy the inequality $D_n \le M_n$. Let us show that, in fact, they coincide; i.e., (1.6) holds. Let us use generalized translation operator (3.7). Assume that $f \in \mathscr{P}_n = \mathscr{P}_n(\mathbb{R})$ and the uniform norm of f is attained at some point $t \in [-1,1]$. It is seen from definition (3.7) that the function $g(x) = (\Theta_t f)$ also is a polynomial of degree n such that $g(1) = f(t)$. Using inequality (1.4), Theorem 3, and Lemma 6, we obtain

$$\|f\|_C = |f(t)| = |g(1)| \le D_n \|g\|_{L_q^\alpha} \le D_n \|f\|_{L_q^\alpha}. \tag{4.1}$$

Since $f \in \mathscr{P}_n$ is an arbitrary function, this implies the inequality $M_n \le D_n$. Relation (1.6) is proved.

Recall that ϱ_n denotes the polynomial that solves problem (1.5). By Theorem 2, this is the unique extremal polynomial in inequality (1.4). We have

$$D_n \,\|\varrho_n\|_{L_q^\phi} = |\varrho_n(1)| \le \|\varrho_n\|_C \le M_n \,\|\varrho_n\|_{L_q^\phi}.$$

In view of (1.6), this implies that

$$\|\varrho_n\|_C = |\varrho_n(1)|$$

and the polynomial ϱ_n is extremal in inequality (1.1).

It remains to verify that, if $\alpha > -1/2$, then ϱ_n is the unique extremal polynomial in inequality (1.1). If the uniform norm of a polynomial f_n extremal in inequality (1.1) is attained at one of the end-points ± 1 of the segment, then the polynomial $f_n(\pm x)$ is extremal in inequality (1.4). By Theorem 2, such polynomial, up to a constant factor, coincides with ϱ_n.

Let us ensure that the uniform norm of none of polynomials extremal in inequality (1.1) can be attained on the interval $(-1,1)$. On the contrary, suppose that the uniform norm of a polynomial $f_n \in \mathscr{P}_n$ extremal in inequality (1.1) is attained at a point $t \in (-1,1)$. For the polynomial f_n, both inequalities and, in particular, the second inequality in (4.1) must turn into equalities. This means that the norm of the operator Θ_t is attained at the polynomial f_n. By Theorem 3, the polynomial f_n is an identical constant for $1 < q < \infty$ and maintains sign on $(-1,1)$ for $q = 1$. Now, it is important that the polynomial f_n maintains sign on $(-1,1)$ in both cases. By formula (3.13), the polynomial $g_n = \Theta_t f_n$ also maintains sign on $(-1,1)$.

The first inequality in (4.1) must turn into an equality at the polynomial f_n. Consequently, the polynomial $g_n = \Theta_t f_n$ is extremal in inequality (1.4). By the uniqueness property of an extremal polynomial, g_n differs from ϱ_n only by a constant factor. The polynomial ϱ_n has n sign changes on $(-1,1)$. Therefore, g_n cannot maintain sign on $(-1,1)$. This contradiction shows that, in the case $\alpha > -1/2$, the uniform norm of a polynomial extremal in inequality (1.1) cannot be attained on the interval $(-1,1)$. Theorem 1 is proved. □

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