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The Bernstein–Szegö Inequality for Fractional Derivatives of Trigonometric Polynomials

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Abstract—On the set $\mathscr{F}_n$ of trigonometric polynomials of degree $n \geq 1$ with complex coefficients, we consider the Szegö operator D_θ^α defined by the relation $D_\theta^\alpha f_n(t) = \cos\theta\, D^\alpha f_n(t) - \sin\theta\, D^\alpha \widetilde{f}_n(t)$ for $\alpha, \theta \in \mathbb{R}$, where $\alpha \geq 0$. Here, $D^\alpha f_n$ and $D^\alpha \widetilde{f}_n$ are the Weyl fractional derivatives of (real) order α of the polynomial f_n and of its conjugate $\widetilde{f}_n$. In particular, we prove that, if $\alpha \geq n \ln 2n$, then, for any $\theta \in \mathbb{R}$, the sharp inequality $\|\cos\theta\, D^\alpha f_n - \sin\theta\, D^\alpha \widetilde{f}_n\|_{L_p} \leq n^\alpha \|f_n\|_{L_p}$ holds on the set $\mathscr{F}_n$ in the spaces L_p for all $p \geq 0$. For classical derivatives (of integer order $\alpha \geq 1$), this inequality was obtained by Szegö in the uniform norm ($p = \infty$) in 1928 and by Zygmund for $1 \leq p < \infty$ in 1931–1935. For fractional derivatives of (real) order $\alpha \geq 1$ and $1 \leq p \leq \infty$, the inequality was proved by Kozko in 1998.
Keywords: trigonometric polynomial, Weyl fractional derivative, Bernstein inequality, Szegö inequality.

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1. HISTORY. AUXILIARY STATEMENTS

1.1. Notation. Let $\mathscr{F}_n = \mathscr{F}_n(\mathbb{P})$ be the set of trigonometric polynomials

$$f_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt) \tag{1.1}$$

of degree $n \geq 1$ with coefficients from the field of real numbers $\mathbb{P} = \mathbb{R}$ or from the field of complex numbers $\mathbb{P} = \mathbb{C}$. The polynomial $\widetilde{f}_n(t) = \sum_{k=1}^n (a_k \sin kt - b_k \cos kt)$ is called the conjugate of the polynomial f_n.

On the set $\mathscr{F}_n(\mathbb{C})$, consider the functional $\|f\|_p = \|f\|_{L_p}$ defined for $0 \leq p \leq +\infty$ by the relations

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^p dt \right)^{1/p}, \qquad 0 < p < \infty,$$

$$\|f\|_\infty = \lim_{p \to +\infty} \|f\|_p = \max\{|f(t)| \colon t \in \mathbb{R}\} = \|f\|_{C_{2\pi}},$$

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$$\|f\|_0 = \lim_{p\to+0} \|f\|_p = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \ln|f(t)|dt\right).$$

1.2. The Bernstein and Szegö inequalities for classical derivatives in the uniform norm. In the set $\mathscr{F}_n(\mathbb{C})$, the following known Bernstein inequality holds:

$$\|f_n'\|_{C_{2\pi}} \le n\|f_n\|_{C_{2\pi}}, \qquad f_n \in \mathscr{F}_n(\mathbb{C}); \tag{1.2}$$

all its extremal polynomials have the form

$$a\cos nt + b\sin nt, \qquad a, b \in \mathbb{C}. \tag{1.3}$$

Bernstein obtained inequality (1.2) for polynomials with real coefficients [1, Sect. 10]. Note that, in the original variant [2, Sect. 12] of paper [1], he proved this inequality with the constant n for odd and even trigonometric polynomials and, as a consequence, with the constant $2n$ in the class of all polynomials (1.1) from $\mathscr{F}_n(\mathbb{R})$. Bernstein writes in his comments [3, Sect. 3.4] to paper [1] that, soon after the appearance of [2], E. Landau communicated to him that inequality (1.2) for polynomials in general form (1.1) (with real coefficients) is an elementary consequence of the inequality for odd polynomials; the proof was first published in [4, Sect. 10].

In 1914, Riesz [5, Sect. 2; 6, Sect. 2] obtained inequality (1.2) with the best constant n (both on the set $\mathscr{F}_n(\mathbb{R})$ and on the set $\mathscr{F}_n(\mathbb{C})$) with the help of the known interpolation formula for the derivative of a trigonometric polynomial; in 1928, Szegö obtained [7] a more general result, which will be given in Theorem B below.

As a consequence of (1.2), the following sharp inequality holds for any natural n and r:

$$\|f_n^{(r)}\|_{C_{2\pi}} \le n^r\|f_n\|_{C_{2\pi}}, \qquad f_n \in \mathscr{F}_n(\mathbb{C}). \tag{1.4}$$

Later, inequalities (1.2) and (1.4) were generalized in different directions. In 1928, Szegö proved the following assertion [7, formulas (1) and (1′)] (see also [8, Vol. 2, Ch. 10, Sect. 3]).

Theorem A. *For any $n \ge 1$ and any real θ, the inequality*

$$\left\|f_n'\cos\theta - \widetilde{f}_n'\sin\theta\right\|_\infty \le n\|f_n\|_\infty, \qquad f_n \in \mathscr{F}_n(\mathbb{R}), \tag{1.5}$$

and, as a consequence, the inequality

$$\left\|\sqrt{\left(f_n'\right)^2 + \left(\widetilde{f}_n'\right)^2}\,\right\|_\infty \le n\|f_n\|_\infty, \qquad f_n \in \mathscr{F}_n(\mathbb{R}), \tag{1.6}$$

hold on the set $\mathscr{F}_n(\mathbb{R})$. Inequalities (1.5) *and* (1.6) *are sharp and turn into equalities only for polynomials* (1.3) *with coefficients $a, b \in \mathbb{R}$.*

Szegö obtained inequality (1.5) with the help of an interpolation formula that generalizes the Riesz formula [5, 6]. More exactly, Szegö proved the following assertion [7, formula (10)] (see also the proof in [8, Vol. 2, Ch. 10, Sect. 3]).

Theorem B. *For $n \ge 1$ and any real θ, the following formula holds on the set of trigonometric polynomials $\mathscr{F}_n(\mathbb{C})$:*

$$f_n'(t)\cos\theta - \widetilde{f}_n'(t)\sin\theta = \sum_{k=1}^{2n} \mu_k f_n(t + t_k), \qquad t \in (-\infty, \infty), \tag{1.7}$$

where

$$t_k = t_k(\theta) = \frac{2k-1}{2n}\pi + \frac{\theta}{n}, \qquad \mu_k = \mu_k(\theta) = \frac{(-1)^{k+1} + \sin\theta}{4n\sin^2(t_k/2)}.$$

Szegö proved [7] formula (1.7) on the set $\mathscr{F}_n(\mathbb{R})$ of real polynomials. Due to linearity, (1.7) also holds for polynomials $f_n \in \mathscr{F}_n(\mathbb{C})$ with complex coefficients. The coefficients of (1.7) satisfy [7, formula (11)] the equality $\sum_{k=1}^{2n} |\mu_k| = n$; therefore, (1.7) implies inequality (1.5) both on the set $\mathscr{F}_n(\mathbb{R})$ and on $\mathscr{F}_n(\mathbb{C})$.

1.3. The Bernstein and Szegö inequalities for fractional derivatives in the uniform norm. The Weyl derivative (or the fractional derivative) of real order $\alpha \geq 0$ of a polynomial f_n written in form (1.1) is the polynomial

$$D^\alpha f_n(t) = \sum_{k=1}^{n} k^\alpha \Big(a_k \cos\Big(kt + \frac{\alpha\pi}{2}\Big) + b_k \sin\Big(kt + \frac{\alpha\pi}{2}\Big)\Big). \tag{1.8}$$

If α is a positive integer, then the fractional derivative coincides with the classical derivative: $D^\alpha f_n = f_n^{(\alpha)}$. Denote by B_n^α the best constant in the Bernstein inequality

$$\|D^\alpha f_n\|_\infty \leq B_n^\alpha \|f_n\|_\infty, \qquad f_n \in \mathscr{F}_n(\mathbb{R}), \tag{1.9}$$

for fractional derivatives on the set $\mathscr{F}_n(\mathbb{R})$. Lizorkin [9, Theorem 2] proved that, if $\alpha \geq 1$, then $B_n^\alpha = n^\alpha$; i.e., an analog of inequality (1.4) holds for fractional derivatives of order $\alpha \geq 1$. Bang [10], Geisberg [11] (see [12, Theorem 19.10 and comments to Sect. 19, Subsect. 8]), and Wilmes [13, Remark 4] studied inequality (1.9) for $0 < \alpha < 1$. The best current estimates [13] are

$$n^\alpha \leq B_n^\alpha \leq 2^{1-\alpha} n^\alpha, \qquad 0 < \alpha < 1.$$

Kozko [14, Corollary 1] extended Theorem A to fractional derivatives (1.8); more exactly, he proved the following assertion for fractional derivatives.

Theorem C. *For any $n \geq 1$, arbitrary real $\alpha \geq 1$, and any real θ, the inequality*

$$\max_{t\in[0,2\pi]} \Big|D^\alpha f_n(t)\cos\theta - D^\alpha \widetilde{f}_n(t)\sin\theta\Big| \leq n^\alpha \|f_n\|_\infty, \qquad f_n \in \mathscr{F}_n(\mathbb{R}), \tag{1.10}$$

and, as a consequence, the inequality

$$\bigg\|\sqrt{\big(D^\alpha f_n\big)^2 + \big(D^\alpha \widetilde{f}_n\big)^2}\bigg\|_\infty \leq n^\alpha \|f_n\|_\infty, \qquad f_n \in \mathscr{F}_n(\mathbb{R}), \tag{1.11}$$

hold on the set $\mathscr{F}_n(\mathbb{R})$.

Let $C_n^\alpha(\theta)$ and C_n^α be the best (i.e., the smallest possible) constants in the inequalities

$$\max_{t\in[0,2\pi]} \Big|D^\alpha f_n(t)\cos\theta - D^\alpha \widetilde{f}_n(t)\sin\theta\Big| \leq C_n^\alpha(\theta) \|f_n\|_\infty, \qquad f_n \in \mathscr{F}_n(\mathbb{R}), \tag{1.12}$$

$$\bigg\|\sqrt{\big(D^\alpha f_n\big)^2 + \big(D^\alpha \widetilde{f}_n\big)^2}\bigg\|_\infty \leq C_n^\alpha \|f_n\|_\infty, \qquad f_n \in \mathscr{F}_n(\mathbb{R}). \tag{1.13}$$

Inequality (1.9) is a special case of (1.12); more exactly, $B_n^\alpha = C_n^\alpha(0)$.

The statements of Theorem C mean that, if $\alpha \geq 1$, then $C_n^\alpha = C_n^\alpha(\theta) = n^\alpha$ for any $\theta \in \mathbb{R}$. It is natural to ask the question about conditions on the parameters under which the values $C_n^\alpha(\theta)$

and C_n^α are equal to n^α. The polynomial $f_n(t)=\cos nt$ shows that $C_n^\alpha \ge C_n^\alpha(\theta)\ge n^\alpha$ for any values of the parameters. Consequently, the fact that inequality (1.10) or (1.11) does not hold means that the constant in corresponding inequality (1.12) or (1.13) is greater than n^α. The following assertion was proved by Kozko [14, Theorem 3] for even n; in the general case, this asssertion was proved in [15, Lemma 3] by a different argument.

Theorem D. *For $n\ge 2$, $0<\alpha<1$, and $\theta=-\alpha\pi/2$, the best constant in inequality* (1.12) *satisfies the strict inequality $C_n^\alpha(\theta)>n^\alpha$.*

Theorem D implies that, for any $n\ge 2$ and $0<\alpha<1$, inequality (1.11) does not hold; more exactly, the best constant C_n^α in (1.13) has the property $C_n^\alpha>n^\alpha$. The exact values of $C_n^\alpha(\theta)$ for $0\le\alpha<1$ are known only in particular cases (see references in [15, 16]).

To prove the results of Theorem C, Kozko [14, Lemma] constructed for the operator

$$D_\theta^\alpha f_n(t)=D^\alpha f_n(t)\cos\theta-D^\alpha\widetilde{f}_n(t)\sin\theta \tag{1.14}$$

a quadrature formula generalizing the quadrature formulas by Riesz [5, 6] and Szegö [7]. This formula has the form

$$D_\theta^\alpha f_n(t)=\sum_{k=0}^{2n-1}\mu_k(\alpha,\theta)(-1)^k f_n(t_k+t),\qquad t_k=\frac{\pi k}{n}+\frac{\alpha\pi}{2n}+\frac{\theta}{n}; \tag{1.15}$$

here,

$$\mu_k(\alpha,\theta)=\Bigg((-1)^{k+1}\sum_{\ell=1}^{n-1}((\ell+1)^\alpha-2\ell^\alpha+(\ell-1)^\alpha)\cos\Big(\ell t_k-\frac{\alpha\pi}{2}-\theta\Big)$$
$$+\,n^\alpha-(n-1)^\alpha+(-1)^{k+1}\cos\Big(\frac{\alpha\pi}{2}+\theta\Big)\Bigg)\Big(4n\sin^2\frac{t_k}{2}\Big)^{-1}$$

in the case $2k+\alpha+2\theta/\pi\ne 0\,(\mathrm{mod}\ 4n)$ and

$$\mu_k(\alpha,\theta)=\frac{1}{n}\Bigg(\sum_{\ell=1}^{n-1}\ell^\alpha+\frac{n^\alpha}{2}\Bigg)$$

in the case $2k+\alpha+2\theta/\pi=0\,(\mathrm{mod}\ 4n)$. For $\alpha\ge 1$, the coefficients $\mu_k(\alpha,\theta)$ of formula (1.15) are nonnegative and $\sum_{k=0}^{2n-1}\mu_k(\alpha,\theta)=n^\alpha$.

Formula (1.14) is valid for polynomials $f_n\in\mathscr{F}_n(\mathbb{C})$ with complex coefficients. Therefore, for any $n\ge 1$, arbitrary real $\alpha\ge 1$, and any real θ, inequality (1.10) actually holds on the set of polynomials $\mathscr{F}_n(\mathbb{C})$; in this case, it turns into an equality only for polynomials (1.3).

1.4. The Bernstein and Szegö inequalities for fractional derivatives in the classical integral norms. Kozko's paper [14, Theorem 1] contains the following assertion.

Theorem E. *Suppose that a function φ is nondecreasing and convex* (*downwards*) *on the semiaxis* $[0,\infty)$. *Then, for any $n\ge 1$, arbitrary real $\alpha\ge 1$, and any real θ, the following inequality holds on the set $\mathscr{F}_n(\mathbb{C})$*:

$$\int_0^{2\pi}\varphi\Big(\Big|D^\alpha f_n(t)\cos\theta-D^\alpha\widetilde{f}_n(t)\sin\theta\Big|\Big)\,dt\le\int_0^{2\pi}\varphi(n^\alpha|f_n(t)|)\,dt,\qquad f_n\in\mathscr{F}_n(\mathbb{C}).$$

This inequality is sharp and turns into an equality for polynomials (1.3)*. If the function* φ *is* (*strictly*) *increasing on* $[0,\infty)$*, then only such polynomials are extremal.*

The function $\varphi(u) = u^p$ for $1 \le p < \infty$ satisfies the conditions of Theorem E; therefore, the following assertion holds as a special case of the theorem.

Corollary 1. *For all* $n \ge 1$, $\alpha \ge 1$, $\theta \in \mathbb{R}$, *and* $p \in [1,\infty)$, *the inequality*

$$\left\| D^\alpha f_n \cos\theta - D^\alpha \widetilde{f}_n \sin\theta \right\|_p \le n^\alpha \|f_n\|_p, \qquad f_n \in \mathscr{F}_n(\mathbb{C}),$$

and, in particular, the inequalities

$$\|D^\alpha f_n\|_p \le n^\alpha \|f_n\|_p, \qquad f_n \in \mathscr{F}_n(\mathbb{C}),$$

$$\left\| D^\alpha \widetilde{f}_n \right\|_p \le n^\alpha \|f_n\|_p, \qquad f_n \in \mathscr{F}_n(\mathbb{C}), \tag{1.16}$$

hold. All three inequalities are sharp and turn into equalities only for polynomials (1.3)*.*

The statements of Theorem E and Corollary 1 for classical derivatives of integer order $\alpha \ge 1$ were established earlier by Zygmund [8, Vol. 2, Ch. 10].

1.5. The Bernstein and Szegö inequalities for classical derivatives in the spaces L_p, $0 \le p < 1$. Let $\Phi^+ = \Phi^+(0,\infty)$ be the class of functions φ defined on $(0,\infty)$ and representable in the form $\varphi(u) = \psi(\ln u)$, where the function $\psi(v) = \varphi(e^v)$ is continuous, nondecreasing, and convex on $(-\infty,\infty)$. The class Φ^+ includes, for example, all nondecreasing convex functions, and the functions u^p for $p > 0$, $\ln u$, $\ln^+ u = \max\{0, \ln u\}$, and $\ln(1+u^p)$ for $p > 0$. Taking into account the properties of convex functions, we can assert that a function φ defined on $(0,\infty)$ belongs to the class Φ^+ if and only if the function $u\varphi'(u)$ is nondecreasing on $(0,\infty)$. The class of functions $\Phi^+ = \Phi^+(0,\infty)$ was introduced in [17, 18], where the Bernstein inequality and its generalizations in the spaces L_p for $p \in [0,1)$ (and more general spaces) were studied. In [19], it was shown that the use of this class is natural in this research area.

In [15, Lemma 1], another description of the class Φ^+ is given. More exactly, a function φ defined on the semiaxis $(0,\infty)$ belongs to the class Φ^+ if and only if it has a finite or equal to $-\infty$ right-hand limit $c = \lim\limits_{r\to+0} \varphi(r)$ at the point 0 and, under the extension $\varphi(0) = c$, the function $\phi(z) = \varphi(|z|)$ is subharmonic in the complex plane $\mathbb{C}$. The following assertion [18, Corollary 6] was proved without using any quadrature formulas.

Theorem F. *For functions* $\varphi \in \Phi^+$ *and any integer* $n \ge 1$ *and* $r \ge 1$*, the following sharp inequality holds*:

$$\int_0^{2\pi} \varphi(|f_n^{(r)}(t)|)dt \le \int_0^{2\pi} \varphi(n^r|f_n(t)|)dt, \qquad f_n \in \mathscr{F}_n(\mathbb{C}). \tag{1.17}$$

Inequality (1.17) *turns into an equality for polynomials* $f_n(t) = ae^{-int} + be^{int}$, *where* $a, b \in \mathbb{C}$. *If the function* $u\varphi'(u)$ *is strictly increasing on* $(0,+\infty)$, *then there are no other extremal polynomials.*

Corollary 2. *For* $0 \le p \le \infty$ *and integer* $n, r \ge 1$, *the inequality*

$$\|f_n^{(r)}\|_p \le n^r \|f_n\|_p, \qquad f_n \in \mathscr{F}_n(\mathbb{C}), \tag{1.18}$$

holds. This inequality is sharp and turns into an equality only for polynomials (1.3)*.*

The functions $\varphi(u) = \ln u$ and $\varphi(u) = u^p$ for $0 \le p < \infty$ satisfy the conditions of Theorem F. Therefore, the statement of Corollary 2 for $0 \le p < \infty$ is contained in Theorem F. Inequality (1.18) for $p = \infty$ is Bernstein inequality (1.4). Inequality (1.18) for $1 \le p < \infty$ was proved by Zygmund [8, Vol. 2, Ch. 10]. Thus, there are at least two proofs of (1.18) for $1 \le p < \infty$.

By (1.16), along with inequality (1.18), the (sharp) inequality

$$\|\widetilde{f}_n^{(r)}\|_{L_p} \le n^r \|f_n\|_{L_p}, \qquad f_n \in \mathscr{F}_n(\mathbb{C}), \tag{1.19}$$

holds for any positive integer n and r and for $1 \le p \le \infty$. As shown in [20, Theorems 3 and 5], generally speaking, inequality (1.19), in contrast to (1.18), cannot be extended to the case $0 \le p < 1$. More exactly, if $r \ge n \ln 2n$, then inequality (1.19) holds for all $p \ge 0$. For a fixed r, the best constant in the analog of inequality (1.19) in the space L_0 behaves as 4^{ε_n} as $n \to \infty$, where $\varepsilon_n = n + o(n)$. It is seen that the growth of this constant with respect to n is essentially greater than that of the constant n^r in (1.19) for $1 \le p \le \infty$.

1.6. The Bernstein–Szegö inequality for fractional derivatives in the spaces L_p, $0 \le p < 1$. For $\theta \in \mathbb{R}$ and $\alpha \ge 0$, consider the Szegö operator on $\mathscr{F}_n(\mathbb{C})$:

$$\begin{aligned} D_\theta^\alpha f_n(t) &= \cos\theta\, D^\alpha f_n(t) - \sin\theta\, D^\alpha \widetilde{f}_n(t) \\ &= \sum_{k=1}^{n} k^\alpha \Big(a_k \cos\Big(kt + \frac{\alpha\pi}{2} + \theta\Big) + b_k \sin\Big(kt + \frac{\alpha\pi}{2} + \theta\Big)\Big). \end{aligned} \tag{1.20}$$

In the present paper, we are primarily interested in the inequality

$$\|D_\theta^\alpha f_n\|_p \le C_n^\alpha(\theta)_p\, \|f_n\|_p, \qquad f_n \in \mathscr{F}_n(\mathbb{C}),$$

with the smallest possible constant $C_n^\alpha(\theta)_p$ for $0 \le p \le \infty$. As said above, Kozko [14] proved that, in the case $\alpha \ge 1$,

$$C_n^\alpha(\theta)_p = n^\alpha \tag{1.21}$$

for all $n \ge 1$ and $1 \le p \le \infty$.

Let us list known order results for the growth of the value $C_n^\alpha(\theta)_p$ as $n \to \infty$ in the case of the other parameters fixed. Belinsky and Liflyand [21] and Runovski and Schmeisser [22, Theorems 5.3 and 5.4] proved that $C_n^\alpha(\theta)_p \asymp n^\alpha$ for $\theta \in \mathbb{R}$, $p > 0$, and $\alpha > (1/p - 1)_+ = \max\{0, 1/p - 1\}$. Belinsky and Liflyand [21] also established that $C_n^\alpha(0)_p \asymp n^{1/p-1}$ for $0 < p < 1$, $0 < \alpha < 1/p - 1$, and $\alpha \notin \mathbb{N}$ and $C_n^\alpha(0)_p \asymp n^{1/p-1} \log^{1/p} n$ for $0 < p < 1$ and $\alpha = 1/p - 1 \notin \mathbb{N}$. For $\alpha = 0$, Kozko [14, Theorem 5] proved that $C_n^0(0)_p \asymp n^{(1/p-1)_+}$ for all $p > 0$ and Leont'eva [23, Theorem 1] proved that $C_n^0(0)_0 \asymp 4^n n^{-1/2}$. In Zygmund's monograph [8, Vol. 1, Ch. 2, Sect. 12], it was shown that $C_n^0(\pi/2)_\infty \asymp \log n$; Taikov [24] found the exact value of $C_n^0(\pi/2)_\infty$.

Let us define $\alpha(n)$ for $n \ge 1$ by the relations

$$\alpha(1) = 0; \qquad \alpha(n) = \frac{\ln 2n}{\ln(n/(n-1))}, \qquad n \ge 2.$$

In what follows, the condition

$$\alpha \ge \alpha(n) \tag{1.22}$$

plays an important role. We have $1/n < \ln(n/(n-1)) < 1/(n-1)$. Therefore, condition (1.22) will hold if the constraint $\alpha \ge n \ln(2n)$ holds, which is clearer (and rather close).

The following assertion is the main result of the present paper.

Theorem 1. *If the order* α *of a fractional derivative satisfies condition* (1.22), *then the sharp inequality*

$$\int_0^{2\pi} \varphi\left(\left|D^\alpha f_n(t)\cos\theta - D^\alpha \widetilde{f}_n(t)\sin\theta\right|\right) dt \le \int_0^{2\pi} \varphi\left(n^\alpha\ |f_n(t)|\right) dt, \qquad f_n \in \mathscr{F}_n(\mathbb{C}), \tag{1.23}$$

holds for any function $\varphi \in \Phi^+$ *and any real* θ. *This inequality turns into an equality for polynomials* $a\cos nt + b\sin nt$, *where* $a, b \in \mathbb{C}$; *if the function* $u\varphi'(u)$ *is strictly increasing on* $(0, +\infty)$, *then there are no other extremal polynomials.*

As a special case of Theorem 1, the following assertion is valid.

Theorem 2. *If the order* α *of a fractional derivative satisfies condition* (1.22), *then the sharp inequality*

$$\left\|D^\alpha f_n \cos\theta - D^\alpha \widetilde{f}_n \sin\theta\right\|_p \le n^\alpha \|f_n\|_p, \qquad f_n \in \mathscr{F}_n(\mathbb{C}), \tag{1.24}$$

holds for $0 \le p < \infty$ *and arbitrary real* θ. *Inequality* (1.24) *turns into an equality for polynomials* $a\cos nt + b\sin nt$, *where* $a, b \in \mathbb{C}$; *there are no other extremal polynomials for* $0 < p < \infty$.

According to Theorem 2, for $n \ge 1$, values of the parameter α satisfying condition (1.22), and any real θ, equality (1.21) also holds in the case $0 \le p < 1$.

Inequality (1.24) for $\theta = 0$, i.e., the inequality

$$\|D^\alpha f_n\|_p \le n^\alpha \|f_n\|_p, \qquad f_n \in \mathscr{F}_n(\mathbb{C}), \tag{1.25}$$

is of interest. In accordance with (1.4), this inequality holds for classical derivatives for any positive integer α, i.e., for $\alpha \ge 1$. One might expect that it holds for all real $\alpha \ge 1$. However, this fact is not valid. According to Theorem 2, inequality (1.25) certainly holds under condition (1.22). Computer calculations allow us to conjecture that inequalities (1.24) and (1.25) hold for real α if and only if the constraint $\alpha \ge 2(n-1)$ holds, which is weaker than (1.22). This conjecture is proved for $n = 2$ in the last section of the present paper.

2. REDUCTION TO PROBLEMS FOR ALGEBRAIC POLYNOMIALS ON THE UNIT CIRCLE OF THE COMPLEX PLANE

The formula

$$f_n(t) = e^{-int} P_{2n}(e^{it}) \tag{2.1}$$

establishes a one-to-one correspondence between the set $\mathscr{F}_n(\mathbb{C})$ of trigonometric polynomials of degree n and the set $\mathscr{P}_{2n}$ of algebraic polynomials of degree $2n$ (see, for example, [8, Vol. 2, Ch. 10]). Using this fact, we can rewrite inequality (1.23) for trigonometric polynomials in the form of the corresponding inequality for algebraic polynomials (on the unit circle of the complex plane). These inequalities are the subject of our study in this section.

2.1. The operation of Szegö composition on the set of algebraic polynomials. Let $\mathscr{P}_n = \mathscr{P}_n(\mathbb{C})$ be the set of algebraic polynomials of degree (at most) $n \ge 1$ with complex coefficients. On the set $\mathscr{P}_n$, consider the functional $\|P_n\|_p = \|P_n\|_{H_p}$ defined by the following relations depending on the value of the parameter p:

$$\|P_n\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|P_n\left(e^{it}\right)\right|^p dt\right)^{1/p}, \qquad 0 < p < \infty,$$

$$\|P_n\|_\infty = \lim_{p\to+\infty} \|P_n\|_p = \max\left\{\left|P_n\left(e^{it}\right)\right| : t \in \mathbb{R}\right\},$$

$$\|P_n\|_0 = \lim_{p\to+0} \|P_n\|_p = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \ln\left|P_n\left(e^{it}\right)\right| dt\right).$$

For polynomials

$$\Lambda_n(z) = \sum_{k=0}^{n} \lambda_k \binom{n}{k} z^k, \qquad P_n(z) = \sum_{k=0}^{n} a_k \binom{n}{k} z^k, \tag{2.2}$$

the polynomial

$$(\Lambda_n P_n)(z) = \sum_{k=0}^{n} \lambda_k a_k \binom{n}{k} z^k \tag{2.3}$$

is called the Szegö composition of Λ_n and P_n. Properties of the Szegö composition can be found in [26, Sect. 5; 27, Ch. 4], see also [28, 29] and references therein. For fixed Λ_n, Szegö composition (2.3) is a linear operator in $\mathscr{P}_n$. The following assertion [25, Theorem 1] is valid for the Szegö composition of polynomials.

Theorem G. *The inequality*

$$\int_0^{2\pi} \varphi(|(\Lambda_n P_n)(e^{it})|)dt \le \int_0^{2\pi} \varphi(\|\Lambda_n\|_0 \ |P_n(e^{it})|)dt \tag{2.4}$$

is valid for functions $\varphi \in \Phi^+$ and any two polynomials from the set $\mathscr{P}_n$ for any $n \ge 1$.

Inequality (2.4) for the function $\varphi(u) = \ln u$ takes the form

$$\|\Lambda_n P_n\|_0 \le \|\Lambda_n\|_0 \ \|P_n\|_0.$$

This inequality in a slightly different form was proved earlier in [29, Theorem 7]. For any Λ_n, this inequality turns into an equality for polynomials $P_n(z) = c(1+z)^n$, where $c \in \mathbb{C}$.

Let Ω_n^+, Ω_n^-, and $\Omega_n^1 = \Omega_n^+ \bigcap \Omega_n^-$ be the sets of polynomials $\Lambda_n \in \mathscr{P}_n$ all of whose n zeros lie in the unit disk $|z| \le 1$, in the domain $|z| \ge 1$, and on the unit circle, respectively. We set $\Omega_n = \Omega_n^+ \bigcup \Omega_n^-$. By the known Poisson–Jensen formula (see, for example, [26, Sect. 3, Problem 175; 31, Ch. 6, Sect. 4]), we have

$$\|\Lambda_n\|_0 = |\lambda_n|, \qquad \Lambda_n \in \Omega_n^+; \qquad \|\Lambda_n\|_0 = |\lambda_0|, \qquad \Lambda_n \in \Omega_n^-;$$

$$\|\Lambda_n\|_0 = |\lambda_n| = |\lambda_0|, \qquad \Lambda_n \in \Omega_n^1. \tag{2.5}$$

Denote by the same symbols Ω_n^+, Ω_n^-, Ω_n^1, and Ω_n the sets of operators (2.3) generated by polynomials Λ_n from the corresponding classes.

The following assertion is a refinement of Theorem G, though it was obtained earlier (see [18, Theorem 4]).

Theorem H. *For any $n \geq 1$, operator $\Lambda_n \in \Omega_n^1$ and function $\varphi \in \Phi^+$, the following inequality holds on the set $\mathscr{P}_n$:*

$$\int_0^{2\pi} \varphi(|(\Lambda_n P_n)(e^{it})|)dt \leq \int_0^{2\pi} \varphi(c_n\ |P_n(e^{it})|)dt, \qquad P_n \in \mathscr{P}_n, \tag{2.6}$$

$$c_n = \|\Lambda_n\|_0 = |\lambda_n| = |\lambda_0|.$$

Inequality (2.6) *is sharp and turns into an equality for polynomials*

$$P_n(z) = az^n + b, \qquad a, b \in \mathbb{C}. \tag{2.7}$$

Under certain conditions on the polynomial Λ_n and the function φ, there are no other extremal polynomials in (2.6) except for (2.7). For fixed $A \in \mathbb{C}$ and $\phi \in \mathbb{R}$, the polynomial

$$I_n(z) = A(ze^{-i\phi} + 1)^n = Ae^{-in\phi}(z + e^{i\phi})^n = A\sum_{k=0}^{n} z^k \binom{n}{k} e^{-ik\phi} \tag{2.8}$$

generates by formula (2.3) the operator

$$I_n P_n(z) = A\sum_{k=0}^{n} c_k \binom{n}{k} e^{-ik\phi} z^k = AP_n(e^{-i\phi}z), \qquad P_n \in \mathscr{P}_n,$$

for which, obviously, any polynomial is extremal in inequality (2.6). For $n \geq 2$, we associate with the polynomial Λ_n defined in (2.2) the polynomial

$$\underline{\Lambda}_{n-2}(z) = \sum_{k=0}^{n-2} \lambda_{k+1} \binom{n-2}{k} z^k \tag{2.9}$$

of degree $n-2$. The property

$$\underline{\Lambda}_{n-2} \in \Omega_{n-2} \tag{2.10}$$

plays an important role in studying the set of polynomials extremal in inequality (2.6). The following assertion is contained in [18, Theorem 2].

Theorem I. *If a function $\varphi \in \Phi^+$ and a polynomial $\Lambda_n \in \Omega_n^1$ for $n \geq 2$ are such that the function $u\varphi'(u)$ is strictly increasing on $(0, +\infty)$, the polynomial Λ_n does not have form* (2.8), *and condition* (2.10) *holds, then only polynomials* (2.7) *are extremal in* (2.6).

2.2. Representation of operator (1.20) in the form of Szegö composition. Let us check that operator (1.20) can be written in the form of the Szegö composition of algebraic polynomials (of degree $2n$). Along with formula (1.1), a polynomial $f_n \in \mathscr{F}_n(\mathbb{C})$ can be written in the exponential form

$$f_n(t) = \sum_{k=-n}^{n} c_k e^{ikt} = \sum_{k=1}^{n} c_{-k} e^{-ikt} + c_0 + \sum_{k=1}^{n} c_k e^{ikt}. \tag{2.11}$$

The coefficients of representations (1.1) and (2.11) are related as follows:

$$c_{-k} = \frac{a_k + ib_k}{2}, \qquad c_k = \frac{a_k - ib_k}{2}, \qquad 1 \le k \le n; \qquad c_0 = \frac{a_0}{2}.$$

Note that formula (2.11) can be written in the form

$$f_n(t) = e^{-int} P_{2n}(e^{it}), \qquad P_{2n}(z) = \sum_{k=0}^{2n} c_{k-n} z^k; \tag{2.12}$$

this is exactly formula (2.1).

Let us find an expression for the operator of fractional differentiation and the Szegö operator on trigonometric polynomials $f_n \in \mathscr{F}_n(\mathbb{C})$ written in exponential form (2.11). For $k \ge 1$, in accordance with the definition of a conjugate polynomial, for the functions

$$e_k^+(t) = e^{ikt}, \qquad e_k^-(t) = e^{-ikt},$$

we have

$$\widetilde{e_k^+}(t) = -ie^{ikt} = e^{-i\pi/2} e^{ikt}, \qquad \widetilde{e_k^-}(t) = ie^{-ikt} = e^{i\pi/2} e^{-ikt}.$$

Based on formula (1.8), we find

$$(D^\alpha e_k^+)(t) = k^\alpha e^{i\frac{\alpha\pi}{2}} e^{ikt}, \qquad D^\alpha e_k^-(t) = k^\alpha e^{-i\frac{\alpha\pi}{2}} e^{-ikt}.$$

Consequently,

$$\widetilde{f_n}(t) = \sum_{k=1}^{n} c_{-k} e^{i\pi/2} e^{-ikt} + \sum_{k=1}^{n} c_k e^{-i\pi/2} e^{ikt},$$

$$D^\alpha f_n(t) = \sum_{k=1}^{n} k^\alpha e^{-i\frac{\alpha\pi}{2}} c_{-k} e^{-ikt} + \sum_{k=1}^{n} k^\alpha e^{i\frac{\alpha\pi}{2}} c_k e^{ikt}.$$

Hence, it is also easy to obtain an expression for Szegö operator (1.20):

$$D_\theta^\alpha f_n(t) = \sum_{k=1}^{n} k^\alpha e^{-i\left(\theta + \frac{\alpha\pi}{2}\right)} c_{-k} e^{-ikt} + \sum_{k=1}^{n} k^\alpha e^{i\left(\theta + \frac{\alpha\pi}{2}\right)} c_k e^{ikt}. \tag{2.13}$$

Polynomial (2.13) has the form $D_\theta^\alpha f_n(t) = e^{-int} G_{2n}(e^{it})$, where

$$G_{2n}(z) = \sum_{k=1}^{n} c_{-k} k^\alpha e^{-i(\pi\alpha/2+\theta)} z^{n-k} + \sum_{k=1}^{n} c_k k^\alpha e^{i(\pi\alpha/2+\theta)} z^{n+k}.$$

The polynomial G_{2n} is the Szegö composition (see definition (2.3)) of the polynomial

$$\Lambda_{2n}^\alpha(z) = \Lambda_{2n}^{\alpha,\theta}(z) = \sum_{k=1}^{n} k^\alpha \binom{2n}{n-k} e^{-i(\pi\alpha/2+\theta)} z^{n-k} + \sum_{k=1}^{n} k^\alpha \binom{2n}{n+k} e^{i(\pi\alpha/2+\theta)} z^{n+k} \tag{2.14}$$

and the polynomial P_{2n} from (2.12). Then, polynomial (2.12) satisfies the formula

$$D_\theta^\alpha f_n(t) = e^{-int} \left(\Lambda_{2n}^\alpha P_{2n}\right)(e^{it}); \tag{2.15}$$

thus, operator (1.20) is represented in the form of Szegö composition of algebraic polynomials.

Theorem 3. *For any $n \ge 1$ and any real α and θ, functions $\varphi \in \Phi^+$ satisfy the inequality*

$$\int\limits_0^{2\pi} \varphi\left(\left|D^\alpha f_n(t)\cos\theta - D^\alpha \widetilde{f}_n(t)\sin\theta\right|\right) dt \le \int\limits_0^{2\pi} \varphi\left(\|\Lambda_{2n}^\alpha\|_0\, |f_n(t)|\right) dt, \qquad f_n \in \mathscr{F}_n(\mathbb{C}). \tag{2.16}$$

Proof. The inequality

$$\int\limits_0^{2\pi} \varphi(|(\Lambda_{2n}^\alpha P_{2n})(e^{it})|)dt \le \int\limits_0^{2\pi} \varphi\left(\|\Lambda_{2n}^\alpha\|_0\, |P_{2n}(e^{it})|\right) dt, \qquad P_{2n} \in \mathscr{P}_{2n}, \tag{2.17}$$

is valid as a special case Theorem G. By relations (2.15) and (2.1), inequality (2.17) on the set $\mathscr{P}_{2n}$ of algebraic polynomials of degree $2n$ coincides with inequality (2.16) on the set $\mathscr{F}_n(\mathbb{C})$ of trigonometric polynomials of degree n. □

2.3. Studying zeros of polynomial (2.14). Let us formulate sufficient conditions under which all $2n$ zeros of polynomial (2.14) lie on the unit circle $\{z \in \mathbb{C}\colon |z| = 1\}$ or, equivalently, all $2n$ zeros of the polynomial

$$e^{-int}\Lambda_{2n}^\alpha(e^{it}) = \sum_{k=1}^n \binom{2n}{n+k} k^\alpha e^{-i(\pi\alpha/2+\theta)} e^{-ikt} + \sum_{k=1}^n \binom{2n}{n+k} k^\alpha e^{i(\pi\alpha/2+\theta)} e^{ikt}$$

$$= 2\sum_{k=1}^n \binom{2n}{n+k} k^\alpha \cos(kt + \pi\alpha/2 + \theta) \tag{2.18}$$

are real.

Lemma 1. *For $n \ge 1$,*

$$\alpha \ge \alpha(n), \tag{2.19}$$

and, for any real θ, all $2n$ zeros of polynomial (2.14) lie on the unit circle.

To prove Lemma 1, we apply the following assertion by Pólya [30].

Theorem J. *If $\lambda, \mu \in \mathbb{R}$, $\lambda^2 + \mu^2 > 0$, and*

$$0 < a_0 < a_1 < a_2 < \ldots < a_n,$$

then the trigonometric polynomials

$$\lambda\big(a_0 + a_1\cos t + \cdots + a_n\cos nt\big) - \mu\big(a_1\sin t + \cdots + a_n\sin nt\big), \tag{2.20}$$

$$\mu\big(a_0 + a_1\cos t + \cdots + a_n\cos nt\big) + \lambda\big(a_1\sin t + \cdots + a_n\sin nt\big)$$

have only real zeros, which interlace.

Proof of Lemma 1. In the case $n = 1$, the statement of the lemma is obvious.

Let $n \ge 2$. The statement of Lemma 1 is equivalent to the fact that all $2n$ zeros of trigonometric polynomial (2.18) are real. We have

$$\cos\left(kt + \frac{\pi\alpha}{2} + \theta\right) = \cos\left(\frac{\pi\alpha}{2} + \theta\right)\cos kt - \sin\left(\frac{\pi\alpha}{2} + \theta\right)\sin kt.$$

Consequently, polynomial (2.18) has form (2.20) with the coefficients

$$a_0 = 0, \qquad a_k = \binom{2n}{n+k} k^\alpha \tag{2.21}$$

and the values of the parameters

$$\lambda = \cos\Big(\frac{\pi\alpha}{2} + \theta\Big), \qquad \mu = \sin\Big(\frac{\pi\alpha}{2} + \theta\Big).$$

Let us check that, under condition (2.19), coefficients (2.21) increase. Obviously, for $\alpha \geq 0$ and $n > 1$, the relation

$$\frac{a_{k+1}}{a_k} = \Big(\frac{k+1}{k}\Big)^\alpha \frac{(n+k)!(n-k)!}{(n+k+1)!(n-k-1)!} = \Big(\frac{k+1}{k}\Big)^\alpha \frac{(n-k)}{(n+k+1)}$$

decreases with respect to $k \in [1, n-1]$. Therefore, if

$$\frac{a_n}{a_{n-1}} = \Big(\frac{n}{n-1}\Big)^\alpha \frac{1}{2n} > 1 \qquad \text{or, equivalently,} \qquad \alpha > \alpha(n) = \frac{\ln 2n}{\ln(n/(n-1))}, \tag{2.22}$$

then coefficients (2.21) increase with respect to k. Thus, under condition (2.22), polynomial (2.18) satisfies the conditions of Theorem J and, therefore, all its $2n$ zeros are real (and, in addition, they all are different). For $\alpha = \alpha(n)$, in view of continuity, the statement of Lemma 1 can be easily proved by means of, for example, the Hurwitz theorem (see [31, Ch. 4, Sect. 3]). $\square$

2.4. Proof of Theorem 1. First, let us apply the statement of Theorem 3. According to Lemma 1, under condition (2.19), all $2n$ zeros of polynomial (2.14) lie on the unit circle; i.e., $\Lambda_{2n}^\alpha \in \Omega_{2n}^1$. The leading coefficient of polynomial (2.14) is $n^\alpha e^{i(\pi\alpha/2+\theta)}$. By formula (2.5), for the constant in inequality (2.16), the formula $\|\Lambda_{2n}^\alpha\|_0 = n^\alpha$ is valid. Thus, under the conditions of Theorem 1, inequality (2.16) coincides with inequality (1.23).

Under condition (2.19), polynomial (2.14) satisfies the conditions of Theorem H. Therefore, inequality (2.17) turns into an equality for polynomials $c_n z^{2n} + c_{-n}$, where $c_n, c_{-n} \in \mathbb{C}$. Consequently, inequality (1.23) turns into an equality for polynomials

$$c_n e^{int} + c_{-n} e^{-int}, \qquad c_n, c_{-n} \in \mathbb{C}, \tag{2.23}$$

or, equivalently, for polynomials (1.3).

To prove the uniqueness property for polynomials (2.23), we apply Theorem I. For $n \geq 2$, by formula (2.9), the polynomial

$$\Lambda_{2(n-1)}^\alpha(z) = \sum_{k=1}^{n-1} k^\alpha \binom{2(n-1)}{n-k-1} e^{-i(\pi\alpha/2+\theta)} z^{n-k-1} + \sum_{k=1}^{n-1} k^\alpha \binom{2(n-1)}{n+k-1} e^{i(\pi\alpha/2+\theta)} z^{n+k-1} \tag{2.24}$$

corresponds to polynomial (2.14); this is polynomial (2.14) corresponding to the degree $n-1$. It is easy to understand that the right-hand side of condition (2.19) grows with respect to n. Therefore, if condition (2.19) holds, then zeros of polynomial (2.24) lie on the unit circle of the complex plane; i.e., $\Lambda_{2(n-1)}^\alpha \in \Omega_{2(n-1)}$. If $n = 1$, then $\Lambda_{2(n-1)}^\alpha \equiv 0$ and, hence, property (2.10) also holds. Therefore, if the function $u\varphi'(u)$ is increasing on $(0, +\infty)$, then, by Theorem I, for all $n \geq 1$, there are no other extremal polynomials in inequality (1.23) except for (2.23). Theorem 1 is proved. $\square$

3. THE BERNSTEIN–SZEGÖ INEQUALITY FOR POLYNOMIALS OF SECOND DEGREE

3.1. Formulation of results. In this section, we study inequalities (1.23) and (1.24) on the set of trigonometric polynomials of second degree. More exactly, we will give necessary and sufficient conditions on the parameters $\alpha \geq 0$ and θ under which the quantity $C_2^\alpha(\theta)_0 = \|\Lambda_4^\alpha\|_0$ is equal to 2^α. The results of this section are contained in the following two statements.

Lemma 2. *Let*

$$d(\alpha,\theta) = 4^{-\alpha}(4^\alpha - 4)^3 - 108\sin^2\Big(\frac{\pi(\alpha-1)}{2} + \theta\Big), \qquad \alpha \in [1,2], \qquad \theta \in [0,\pi).$$

For any $\theta \in (0,\pi)$, the equation

$$d(\alpha,\theta) = 0$$

has a unique solution $\alpha = \alpha_2(\theta) \in (1,2)$; in addition,

$$d(\alpha,\theta) < 0 \quad \textit{for} \quad \alpha \in [1,\alpha_2(\theta)) \qquad \textit{and} \qquad d(\alpha,\theta) > 0 \quad \textit{for} \quad \alpha \in (\alpha_2(\theta),2]. \tag{3.1}$$

For $\theta = 0$,

$$d(\alpha,0) < 0 \quad \textit{for} \quad \alpha \in (1,2) \qquad \textit{and} \qquad d(1,0) = d(2,0) = 0.$$

Theorem 4. *If $\theta = 0$, then*

$$C_2^\alpha(0)_0 = 2^\alpha \quad \textit{for} \quad \alpha = 1 \textit{ and } \alpha \geq 2, \qquad C_2^\alpha(0)_0 > 2^\alpha \quad \textit{for} \quad \alpha \in [0,1) \cup (1,2).$$

If $\theta \in (0,\pi)$, then

$$C_2^\alpha(\theta)_0 = 2^\alpha \quad \textit{for} \quad \alpha \geq \alpha_2(\theta), \qquad C_2^\alpha(\theta)_0 > 2^\alpha \quad \textit{for} \quad \alpha \in [0,\alpha_2(\theta)).$$

3.2. Proof of Lemma 2. We divide the proof of the lemma into several steps.

(1) First, consider the case $\theta = 0$. In this case,

$$d(\alpha) = d(\alpha,0) = \frac{(4^\alpha-4)^3}{4^\alpha} - 108\sin^2\Big(\frac{\pi(\alpha-1)}{2}\Big).$$

The well-known inequality $\sin x > 2x/\pi$ for $x \in (0,\pi/2)$ implies the estimate

$$d(\alpha) < q(\alpha) = \frac{(4^\alpha-4)^3}{4^\alpha} - 108(\alpha-1)^2, \qquad \alpha \in (1,2).$$

We have

$$q'(\alpha) = 2\ln 4\Big(4^{2\alpha} - 6 \times 4^\alpha + \frac{32}{4^\alpha}\Big) - 216(\alpha-1),$$

$$q''(\alpha) = 4\ln^2 4\Big(4^\alpha(4^\alpha-3) - \frac{16}{4^\alpha}\Big) - 216.$$

It is seen from the latter expression that $q''(\alpha)$ is increasing with respect to $\alpha \in [1,2]$. Consequently, $q'(\alpha)$ is (strictly) convex downwards; moreover, $q'(1) = 0$ and $q'(2) > 0$. Hence, the function $q(\alpha)$ is strictly increasing at the point $\alpha = 2$ and changes the character of monotonicity on the interval $[1,2]$ at most once. However, since $q(1) = q(2) = 0$, we have $d(\alpha) < q(\alpha) < 0$ for $\alpha \in (1,2)$. Thus, we have proved the statement of the lemma for $\theta = 0$.

(2) Let $\theta \in (0, \pi/2)$. Let us introduce the notation

$$g(\alpha) = \frac{(4^\alpha - 4)^3}{4^\alpha}, \qquad h(\alpha,\theta) = 108\sin^2\Big(\frac{\pi(\alpha-1)}{2} + \theta\Big) = h\Big(\alpha + \frac{2\theta}{\pi}, 0\Big).$$

For $\alpha \in [1, 2-2\theta/\pi]$, the function $h(\alpha,\theta)$ is strictly increasing with respect to θ; therefore, we have

$$d(\alpha,\theta) < g(\alpha) - h(\alpha, 0) = d(\alpha, 0) \le 0.$$

For $\alpha \in [2-2\theta/\pi, 2]$, the function $g(\alpha)$ is increasing and $h(\alpha,\theta)$ is decreasing with respect to α and

$$d(2,\theta) = g(2) - h(2,\theta) = 108 - 108\sin^2\Big(\frac{\pi}{2} + \theta\Big) > 0.$$

Consequently, there exists unique $\alpha = \alpha_2(\theta) \in (1, 2)$ with property (3.1). In addition, $\alpha_2(\theta) \in (2 - 2\theta/\pi, 2)$ in this case.

(3) Let $\theta \in [\pi/2, \pi)$. In this case, for $\alpha \in [1, 3 - 2\theta/\pi]$, the function $g(\alpha)$ is increasing and $h(\alpha,\theta)$ is decreasing with respect to α; in addition, $g(1) = 0$ and $h(3-2\theta/\pi, \theta) = 0$. Therefore, there exists unique $\alpha = \alpha_2(\theta) \in (1, 3-2\theta/\pi)$ such that $d(\alpha,\theta) < 0$, $\alpha \in [1, \alpha_2(\theta))$, $d(\alpha,\theta) > 0$, and $\alpha \in (\alpha_2(\theta), 3-2\theta/\pi)$. For $\alpha \in (3-2\theta/\pi, 2)$, we have $\alpha + 2\theta/\pi - 2 \in (1, 2)$; therefore,

$$d(\alpha,\theta) = g(\alpha) - h\Big(\alpha + \frac{2\theta}{\pi} - 2, 0\Big) < g\Big(\alpha + \frac{2\theta}{\pi} - 2\Big) - h\Big(\alpha + \frac{2\theta}{\pi} - 2, 0\Big) \le 0.$$

Thus, there exists unique $\alpha = \alpha_2(\theta) \in (1, 2)$ with property (3.1). In addition, $\alpha_2(\theta) \in (1, 3-2\theta/\pi)$ in this case. The lemma is proved. □

3.3. Proof of Theorem 4. Consider polynomial (2.14) for $n = 2$ and $\alpha > 0$:

$$\Lambda_4^\alpha(z) = 2^\alpha e^{-i(\pi\alpha/2+\theta)} + 4e^{-i(\pi\alpha/2+\theta)}z + 4e^{i(\pi\alpha/2+\theta)}z^3 + 2^\alpha e^{i(\pi\alpha/2+\theta)}z^4 = b_0 + b_1 z + b_3 z^3 + b_4 z^4.$$

The polynomial Λ_4^α has the property $\Lambda_4^\alpha(z) = z^4\overline{\Lambda_4^\alpha(1/\overline{z})}$. Therefore, either all zeros of Λ_4^α lie on the boundary of the disk $B = \{z : |z| < 1\}$ or Λ_4^α has zeros both in the disk B and in the domain $|z| > 1$. If all zeros of Λ_4^α lie on the boundary of B, then $C_2^\alpha(\theta) = 2^\alpha$; otherwise, $C_2^\alpha(\theta) > 2^\alpha$.

Let us study the number of zeros of Λ_4^α in the open disk B. According to Theorem (45,2) from [27], the polynomials Λ_4^α and

$$Q(z) = 4\overline{b_4} + 3\overline{b_3}z + \overline{b_1}z^3 = 4\times 2^\alpha e^{-(\pi\alpha/2+\theta)} + 3\times 4e^{-i(\pi\alpha/2+\theta)}z + 4e^{i(\pi\alpha/2+\theta)}z^3$$

have the same number of zeros in the disk B. Therefore, in what follows, we will study the polynomial Q. For computational convenience, we multiply Q by $e^{i(\pi\alpha/2+\theta)}/4$ and obtain the polynomial

$$Q_1(z) = 2^\alpha + 3z + e^{i(\pi\alpha+2\theta)}z^3 = a_0 + a_1 z + a_2 z^2 + a_3 z^3.$$

Consider the following three cases.

(1) If $\alpha \in [0, 1)$ or $\alpha \ge 2$, then, applying Rouché's theorem, it is easy to see that $Q_1(z)$ has one zero in B for $\alpha \in [0, 1)$ and has no zeros in B for $\alpha \ge 2$.

(2) If $\alpha = 1$, then $Q_1(z) = 2 + 3z + e^{i(\pi+2\theta)}z^3$. By Lemma (42,1) from [27], the number of zeros in the disk B of the polynomial Q_1 is the number of zeros in B of the polynomial

$$Q_2(z) = 3(1 + 2z + e^{i(\pi+2\theta)}z^2).$$

However, if z_1 and z_2 are zeros of Q, then, by the Viète formulas, $z_1 z_2 = -e^{-i(\pi+2\theta)}$ and $z_1 + z_2 = 2e^{-i(\pi+2\theta)}$. Consequently, if $\theta = 0$, then the polynomial Q_2 has no zeros in B, and, if $\theta \in (0, \pi)$, then Q_2 has one zero in the disk B.

(3) It remains to consider the case $\alpha \in (1, 2)$. To find the number of zeros of Q_1 in the disk B, we apply Theorem (43,1) from [27]. If all the determinants

$$\Delta_k = \Delta_k(\alpha,\theta) = \begin{vmatrix} a_0 & 0 & 0 & \dots & 0 & a_n & a_{n-1} & \dots & a_{n-k+1} \\ a_1 & a_0 & 0 & \dots & 0 & 0 & a_n & \dots & a_{n-k+2} \\ & & & \dots & & & & \dots & \\ a_{k-1} & a_{k-2} & a_{k-3} & \dots & a_0 & 0 & 0 & \dots & a_n \\ \overline{a_n} & 0 & 0 & \dots & 0 & \overline{a_0} & \overline{a_1} & \dots & \overline{a_{k-1}} \\ \overline{a_{n-1}} & \overline{a_n} & 0 & \dots & 0 & 0 & \overline{a_0} & \dots & \overline{a_{k-2}} \\ & & & \dots & & & & \dots & \\ \overline{a_{n-k+1}} & \overline{a_{n-k+2}} & \overline{a_{n-k+3}} & \dots & \overline{a_n} & 0 & 0 & \dots & \overline{a_0} \end{vmatrix}, \qquad k = 1, 2, 3,$$

are different from 0, then the number of zeros of the polynomial Q_1 in the disk B is the number of sign changes in the sequence 1, Δ_1, Δ_2, Δ_3. An elementary computation leads to

$$\Delta_1(\alpha,\theta) = 4^\alpha - 1, \qquad \Delta_2(\alpha,\theta) = (4^\alpha - 1)^2 - 9,$$

$$\Delta_3(\alpha,\theta) = 64^\alpha - 12 \times 16^\alpha - 6 \times 4^\alpha - 64 - 27 \times 4^\alpha \times 2 \times \cos(\pi\alpha + 2\theta)$$
$$= 4^\alpha \left(\frac{(4^\alpha - 4)^3}{4^\alpha} - 108 \sin^2\left(\frac{\pi(\alpha - 1)}{2} + \theta \right) \right).$$

We have $\Delta_1 > 0$ and $\Delta_2 > 0$, while the behavior of the sign of Δ_3 is completely described in Lemma 2, which implies the statement of the theorem. $\square$

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