FILTERS BASED ON AGGREGATION OPERATORS. Part 4. Generalized vector median filters

Labunets V.G., Gainanov D.N., Arslanova R. A.[1], Ostheimer E.[2]
[1]*Ural Federal University named after the first President of Russia B.N.Yeltsin*
pr. Mira, 19, Yekaterinburg, Sverdlovsk region, 620002, Russian Federation
ph.: 375-48-48, e-mail: vlabunets05@yahoo.com
[2]*Capricat LLC 1340 S. Ocean Blvd., Suite 209 Pompano Beach 33062 Florida USA*

Abstract — In this work, a new class of nonlinear filters called vector median aggregation filters for multispectral image processing is introduced and applied to the color image filtering problem. These filters are based on an arbitrary pair of aggregation operators and a selection rule. We show that a large body of vector median filters proposed to date constitute a proper subset of new vector median aggregation filters set.

ФИЛЬТРЫ,ОСНОВАННЫЕ НА АГРЕГАЦИОННЫХ ОПЕРАТОРАХ Часть 4. Обобщенные векторные медианные фильтры

Лабунец В.Г., Гайнанов Д.Н., Арсланова Р. А.[1], Остхаймер Е.[2]
[1]*Уральский федеральный университет имени первого Президента России Б.Н. Ельцина*
проспект Мира, 19, Екатеринбург, Свердловская обл.,620002, Россия
тел.: (343) 375-48-48, e-mail: vlabunets05@yahoo.com
[2]*Capricat LLC 1340 S. Ocean Blvd., Suite 209 Pompano Beach 33062 Florida USA*

Аннотация — В данной работе вводится новый класс нелинейных фильтров для обработки мультиспектральных изображений, названный векторными медианными агрегационными фильтрами. Они основываются на произвольной паре агрегационных операторов и одном селекционном правиле. Мы показываем, что большое множество векторных медианных фильтров, предложенных к настоящему времени, являются собственным подмножеством множества новых фильтров.

I. Introduction

Median filtering has been widely used in image processing as an “edge preserving” filter. The basic idea is that the pixel value is replaced by the median of the pixels contained in a window around it. In this work this idea is extended to vector-valued images, based on the fact that the median is also the value that minimizes the L_1 norm between all the pixels in the window. In this case, we need to define a distance between pairs of objects on the domain of application. Let $\langle \mathbf{D}, \rho \rangle$ be a metric spaces, where ρ is a distance function for pairs of objects in $\langle \mathbf{D}, \rho \rangle$ (that is, $\rho : \mathbf{D} \times \mathbf{D} \to \mathbf{R}^+$). Let $w_1, w_2, ..., w_N$ be N weights summing to 1 and let $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N \in \mathbf{D} \subset \mathbf{R}^n$ be N observations from $\langle \mathbf{D}, \rho \rangle$.

Definition 1 [1.2]**.** The weighted Fréchet median or generalized vector-valued median is the point, $\mathbf{c}_{opt} \in \mathbf{D}$, that minimizes the Fréchet function $\sum_{i=1}^{N} w_i \rho\left(\mathbf{c}, \mathbf{x}^i\right)$ (the weighted sum distances from an arbitrary point to each point $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N \in \mathbf{D} \subset \mathbf{R}^K$) and is formally defined as

$$\mathbf{c}_{opt} = \mathbf{Frech}\left(\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N\right) =$$
$$= \mathbf{VectMed}\left(\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N\right) = \mathbf{arg}\min_{\mathbf{c} \in \mathbf{D}} \sum_{i=1}^{N} w_i \rho\left(\mathbf{c}, \mathbf{x}^i\right). \quad (1)$$

Note that **argmin** means the argument for which the sum is minimized. In this case, it is the point $\mathbf{c}_{opt}$ from $\mathbf{D} \subset \mathbf{R}^n$ for one the sum of all distances to the $\mathbf{x}^i$'s is minimum. So, the vector-valued median of a discrete set of sample points in a Euclidean space $\mathbf{R}^K$ is the point minimizing the sum of distances to the sample points. This generalizes the ordinary median, which has the property of minimizing the sum of distances for one-dimensional data, and provides a central tendency higher dimensions. The geometric median can be generalized from Euclidean spaces to general Riemannian using the same idea which is used to define the Fréchet median on a Riemannian manifold [1,2].

Example 1. Arithmetic mean. In the numerical setting, i. e., if the domain D is a subset of the real field $D \subset \mathbf{R}$, when $\rho(x, y) = (x - y)^2$, the aggregation operator results to be the arithmetic mean, i.e. for real numbers, the arithmetic mean is a Fréchet mean, using as distance function the usual Euclidean distance $\rho(x, y) = (x - y)^2$.

Example 2. Geometric mean. On the positive real numbers, we can define the (hyperbolic) distance function $\rho(x, y) = |\log x - \log y|$. The geometric mean is the corresponding Fréchet mean.

Example 3. The Kolmogoov mean. Given an invertible function K, we can define the K-mean as a Fréchet mean by introducing the metric $\rho(a, b) = |K(a) - K(b)|$. This is sometimes called the Kolmogorov mean.

II. Main Part

Let $\mathbf{D}$ be the domain of application (a aggregation space $\langle \mathbf{D}, \rho\text{-}\mathbf{Aggreg} \rangle$), where $d\text{-}\mathbf{Aggreg}$ is an aggre-

gation pseudo-distance function for pairs of objects in $\mathbf{D}$, that is, $d\text{-}\mathbf{Aggreg}:\mathbf{D}\times\mathbf{D}\to\mathbf{R}^{+}$:

$$\rho\text{-}\mathbf{Aggreg}(\mathbf{c},\mathbf{x}) = \\ = \mathbf{Aggreg}(c_1 - x_1, c_2 - x_2, ..., c_n - x_n), \quad (2)$$

where $\mathbf{Aggreg}$ is an arbitrary aggregation operator. Then, the new generalized aggregation operators ${}^{r}\mathbf{OSVectAgg}$ applied to $(\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N)$ are formally defined as

$$^{r}\mathbf{OSVectAgg}(\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N) := \\ = \arg\underset{\mathbf{c}}{{}^{r}\mathbf{OS}}\Big[\mathbf{Aggreg}_2\big(\rho\text{-}\mathbf{Aggreg}_1(\mathbf{c},\mathbf{x}^1), \\ \rho\text{-}\mathbf{Aggreg}_1(\mathbf{c},\mathbf{x}^2), ..., \rho\text{-}\mathbf{Aggreg}_1(\mathbf{c},\mathbf{x}^N)\big)\Big], \quad (3)$$

where r is rank of ordered statistics (${}^{r}\mathbf{OS} = \mathbf{Min}$, ${}^{r}\mathbf{OS} = \mathbf{Med}$, ${}^{r}\mathbf{OS} = \mathbf{Max}$, when $r = 1$, $r = n/2, r = N$, respectively). ${}^{r}\mathbf{OSVectAgg}$ depends on two aggregation operators and a selection rule ${}^{r}\mathbf{OS}$,

$$^{r}\mathbf{OSVectAgg} = \\ = {}^{r}\mathbf{OSVectAgg}\left({}^{r}\mathbf{OS}, \rho\text{-}\mathbf{Aggreg}_1, \mathbf{Aggreg}_2\right) \quad (4)$$

If $\mathbf{Aggreg}_2 = \sum$ and ${}^{r}\mathbf{OS} = \mathbf{Min}$, then

$$^{r}\mathbf{OSVectAgg}(\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^N) := \\ = \arg\min_{\mathbf{c}} \sum_{i=1}^{N} \rho\text{-}\mathbf{Aggreg}(\mathbf{c},\mathbf{x}^i) \quad (5)$$

and if $\rho\text{-}\mathbf{Aggreg}(\mathbf{c},\mathbf{x}^i) = \rho(\mathbf{c},\mathbf{x}^i)$ is a metric distance, then we have the Fréchet median.

Let us introduce the observation model and notion used throughout the paper. We consider noise images of the

$$\vec{\mathbf{f}}_{Mcol}(\mathbf{x}) = \vec{\mathbf{s}}_{Mcol}(\mathbf{x}) + \vec{\boldsymbol{\eta}}_{Mcol}(\mathbf{x}),$$

where $\vec{\mathbf{s}}_{Mcol}(\mathbf{x})$ is the original K-channel image

$$\vec{\mathbf{s}}_{Mcol}(\mathbf{x}) = (s_1(\mathbf{x}), s_2(\mathbf{x}), ..., s_K(\mathbf{x})),$$

$\vec{\boldsymbol{\eta}}_{Mcol}(\mathbf{x})$ denotes the K-channel noise

$$\vec{\boldsymbol{\eta}}_{Mcol}(\mathbf{x}) = (\eta_1(\mathbf{x}), \eta_2(\mathbf{x}), ..., \eta_K(\mathbf{x}))$$

introduced into the image $\vec{\mathbf{s}}_{Mcol}(\mathbf{x})$ to produce the corrupted image $\vec{\mathbf{f}}_{Mcol}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_K(\mathbf{x}))$. Here $\mathbf{x} = (i, j) \in \mathbf{Z}^2$ (or $\mathbf{x} = (i, j, k) \in \mathbf{Z}^3$) is a 2D (or 3D) coordinates that belong to the image domain and represent the pixel location. If $\mathbf{x} \in \mathbf{Z}^2, \mathbf{Z}^3$ then $\vec{\mathbf{f}}_{Mcol}(\mathbf{x})$, $\vec{\mathbf{s}}_{Mcol}(\mathbf{x})$, $\vec{\mathbf{n}}_{Mcol}(\mathbf{x})$ are 2D or 3D multichannel images, respectively.

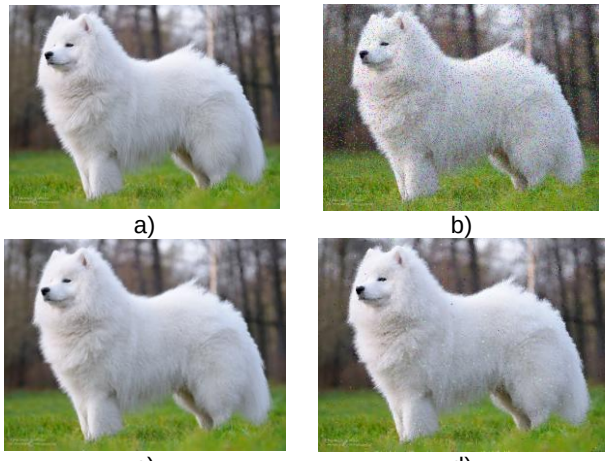
The aim of image enhancement is to reduce the noise as much as possible or to find a method which, given $\vec{\mathbf{s}}_{Mcol}(\mathbf{x})$, derives an image $\hat{\vec{\mathbf{s}}}_{Mcol}(\mathbf{x})$ as close as possible to the original $\vec{\mathbf{s}}_{Mcol}(\mathbf{x})$, subject to a suitable optimality criterion. In a 2D standard linear and median filters with a square window $\left[\mathrm{M}_{(i,j)}(m,n)\right]_{m=-r,n=-r}^{m=+r,n=+r}$ of size $N = (2r+1)\times(2r+1)$ is located at (i, j) the mean replaces the central pixel

$$\hat{\vec{\mathbf{s}}}_{Mcol}(i, j) = \underset{(m,n)\in M_{(i,j)}}{\mathbf{Mean}} \left\{\vec{\mathbf{f}}_{Mcol}(m,n)\right\},$$

where $\hat{\vec{\mathbf{s}}}_{Mcol}(i, j)$ is the filtered image, $\left\{\vec{\mathbf{f}}_{Mcol}(m,n)\right\}_{(m,n)\in M_{(i,j)}}$ is image block of the fixed size N extracted from $\vec{\mathbf{f}}_{Mcol}$ by moving window $\mathrm{M}_{(i,j)}$ at the position (i, j), **Mean** is the mean (average) operators. When those filters are modified as follows

$$\hat{\vec{\mathbf{s}}}_{Mcol}(i, j) = \underset{(k,l)\in M(i,j)}{{}^{r}\mathbf{OSVectAgg}} \left\{\vec{\mathbf{f}}_{Mcol}(k,l)\right\}$$

it becomes vector-valued aggregation filter, where ${}^{r}\mathbf{OSVectAgg}$ is a generalized average (3). In Fig.1 we present two cases: Fig.1c: $\rho\text{-}\mathbf{Aggreg}_1 = \rho_1, \mathbf{Aggreg}_2 = \mathbf{Mean}$, ${}^{r}\mathbf{OS} = \mathbf{Min}$, $\mathrm{PSNR} = 32.2$ $\mathrm{SSIM} = 0.81$ and Fig.1d: $\rho\text{-}\mathbf{Aggreg}_1 = \rho_2$, $\rho\text{-}\mathbf{Aggreg}_1 = \rho_2, \mathbf{Aggreg}_2 = \mathbf{Mean}$, ${}^{r}\mathbf{OS} = \mathbf{Min}$, $\mathrm{PSNR} = 27,4$, $\mathrm{SSIM} = 0,68$.



a) b) c) d)

Fig. 1. Original (a) and noise (b) images. Denoised images (c) and (d).

Рис. 1. Исходное (a) и зашумленное (b) изображения. Отфильтрованные изображения (c) и (d).

III. Conclusion

A new class of nonlinear generalized vector-valued median aggregation filters for multichannel image processing is introduced in this paper. Defined filters depend on finite set of free parameters (two aggregation operators and a selection rule), which could be changed independently of one another. For each set of parameters values we get the unique class of nonlinear filters.

This work was supported by grants the RFBR № 13-07-12168, RFBR № 13-07-00785 and by the MES RF grant №218-03-167 according to the MES RF agreement № 02.G25.31.0055 (12.02. 2013).

IV. References

[1] Bajaj C. Proving geometric algorithms nonsolvability: An application of factoring polynomials. *Jornal of Symbolic Computaton*, 1986, No. 2, pp. 99–102.

[2] Bajaj C. The algebraic degree of geometric optimization problems. *Discrete and Computation Geometry*, 1988, No. 3, pp. 177–191.